Contributions to the theory of sound propagation in ducts with bulk-reacting lining[a)]

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(Received 26 February 1984; accepted for publication 20 November 1984)

A theoretical analysis of sound propagation in cylindrical ducts lined with porous material (bulk absorbers) is presented. Three configurations are discussed. (1) The porous material is homogeneous; then the sound field is built up of modes. (2) The properties of the liner vary slowly in an axial direction. A pure modal solution is, in general, not possible, but the field can be described by modes of the homogeneous liner, of which the amplitudes and wavenumbers vary in an axial direction slowly with the liner (multiple scale solution). (3) The porous material is embedded in an annular structure of partitions. For structural reasons this is a common situation. If the pitch of the partitions is small, the sound field in the liner is, per circumferential mode, decoupled from the duct field, and its effect can be described acoustically by an impedance (per circumferential mode). This "quasi-point-reacting" boundary condition may be easily incorporated in calculation procedures for ducts with locally reacting lining.

PACS numbers: 43.20.Mv, 43.20.Bi

LIST OF SYMBOLS

a	radius of interface duct/liner
A	modal amplitude duct field
b	outer radius duct with liner
B	modal amplitude liner field
c_0	mean sound speed in duct
c_e	effective sound speed in porous material
d	liner thickness $= b - a$
D	ratio modal amplitudes B/A
$E=0$	eigenvalue equation
f	auxiliary function
j'_{mn}	nth zero of J'_m
J_m	mth order Bessel function of the first kind
k	free-field wavenumber $= \omega/c_0$
l	index (integer)
L	width of partition
$\mathscr{L}$	differential operator
m	circumferential eigenvalue (integer)
n	index (integer)
p	acoustic pressure
P	modelike basis function
r	radial coordinate
t	time coordinate
$\mathbf{v}$	acoustic velocity
x	axial coordinate
X	slow variable ϵx
Y_m	mth order Bessel function of the second kind
Z	impedance of cover plate
α	radial wavenumber duct
β	radial wavenumber liner
γ	axial wavenumber
ϵ	small parameter
ζ	ratio radial pressure gradients at interface
θ	polar angle
λ	radial wavenumber in partitioned liner
μ	free-field wavenumber in porous material
ρ	acoustic density
ρ_0	mean density in duct
ρ_e	effective density of porous material
σ	resistivity of porous material
ω	radial frequency
Ω	porosity of porous material

INTRODUCTION

One of the central problems in engineering duct acoustics is how to reduce the noise level in a duct. A usual approach is to apply some sort of lining of acoustically absorbing material. The question is then to specify the liner which produces the maximum damping for a given duct, incoming sound field, and type of liner. If these problem parameters are known in sufficient detail, and allow a mathematical treatment, we can calculate the sound field and hence predict the best liner.

Whether or not a mathematical calculation is possible depends, among other things, on the type of liner. Well known is the honeycomb array of air-filled Helmholtz resonators, e.g., in many aircraft turbo engines.[1] The advantages of this type of lining are a lightweight, solid structure, and mathematically that it is locally reacting and can be represented by an effective impedance of the wall. A disadvantage is its limited frequency range: the damping capacity of the resonator quickly diminishes for off-design frequencies. Therefore, a lining consisting of a layer of porous material (bulk absorber) is sometimes to be preferred since its properties are much less frequency dependent.[2] However, the effect of the liner cannot be described anymore by an effective impedance, since acoustic waves in the liner also propagate

[a)] Paper prepared for presentation at the 106th Meeting of the Acoustical Society of America, 7–11 November 1983, at San Diego, CA.

 0001-4966/85/051681-05$00.80

parallel to the wall, and the duct and liner fields are essentially coupled. Nevertheless, for simple duct geometries (like a cylinder) and homogeneous, isotropic material it is still possible to describe the sound field relatively easily by a modal expansion, as was shown in the classical paper of Scott.[3] The modes are those waveforms which exist in both the duct and the liner, retaining their form as they travel along the duct with constant axial wavenumber and decay ratio. Another approach, as discussed by Bauer,[4] is to consider an array of Helmholtz resonators, filled with porous material. Then the liner is again locally reacting, and its acoustic properties can be described by an effective impedance.

In the present paper we will extend these possibilities to *theoretically* describe the sound field in an infinite cylindrical duct with bulk-absorbent lining with the following two configurations: (1) a liner of which the properties of the porous material vary slowly (relative to the acoustic wavelength) in axial direction; (2) a liner consisting of a segmented structure of annular partitions, with a small pitch, filled with porous material.

The field with the slowly varying liner will be described by a multiple scale solution[5] of slowly varying "constant-liner" modes. This configuration may be considered as a generalization of the classic, homogeneous liner problem,[3] which we will therefore briefly discuss as well. The geometry with annular partitions arises naturally from structural constraints, especially in flow ducts with considerable axial pressure gradients. Although now the duct field and liner field cannot be decoupled in general, we will show that this is possible per circumferential mode, giving rise to an effective, circumferential periodicity-dependent impedance.

The model for sound waves in porous material that our calculations are based on can be found in Nayfeh *et al.*,[1] or Morse and Ingard.[2] In here (and in addition, in Ingard[6]) a more thorough discussion can be found on the various physical backgrounds. The porous material is essentially assumed to be described by its porosity, resistivity, effective density, and effective sound speed, all of which are frequency dependent. To complete the modeling a perforate plate is assumed on top of the porous material. Its thickness is negligible, while its presence will be described mathematically by an impedance, assuming a continuous velocity. Traditionally, such a perforate is acoustically open to utilize the porous material as much as possible, but it is included in the present theory, as an additional degree of freedom which can be taken advantage of.

I. FORMULATION OF THE PROBLEM

Consider in the polar cylindrical coordinate system (x,r,θ) the duct $r = b$. The rigid walls are lined with a layer $a \leqslant r \leqslant b$ of isotropic porous material (bulk-reacting liner) of thickness $d = b - a$, with a perforate plate at the interface $r = a$. The medium in the duct is perturbed by sound waves, varying harmonically in time with (positive) circular frequency ω through a factor $\exp(i\omega t)$. (This factor will be suppressed throughout.) These waves are described by

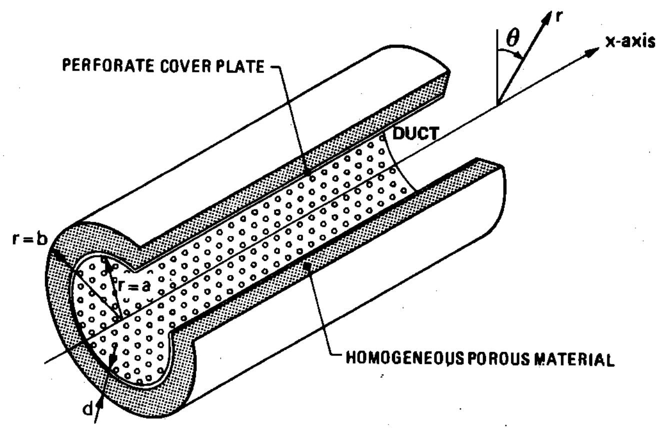

FIG. 1. Homogeneous liner configuration.

$0 \leqslant r \leqslant a$:

$$i\omega\rho + \rho_0 \nabla\cdot\mathbf{v} = 0,$$
$$i\omega\rho_0\mathbf{v} + \nabla p = \mathbf{0}, \quad p = c_0^2\rho, \tag{1}$$

$a \leqslant r \leqslant b$:

$$i\omega\Omega\rho + \rho_0\nabla\cdot\mathbf{v} = 0$$
$$(i\omega\rho_e + \sigma)\mathbf{v} + \nabla p = \mathbf{0}, \quad p = c_e^2\rho, \tag{2}$$

where ρ, p, and $\mathbf{v}$ denote perturbation density, pressure, and velocity, while ρ_0 and c_0 denote mean values of density and sound speed; Ω, σ, ρ_e, and c_e, characterizing the porous material, stand for porosity, resistivity, effective density, and effective (complex) speed of sound, respectively. Note that, in general, these material parameters are frequency dependent; a correct time domain version of Eqs. (2) will therefore involve convolution integrals.

The rigid wall implies a boundary condition of vanishing radial velocity at $r = b$, so

$$\frac{\partial}{\partial r} p(x,b,\theta) = 0. \tag{3}$$

The perforated cylinder at $r = a$ is assumed to induce a pressure jump linear in the continuous, radial velocity, with a complex factor Z. This implies then

$$p(x,a-,\theta) - p(x,a+,\theta) = -\frac{Z}{i\omega\rho_0}\frac{\partial}{\partial r}p(x,a-,\theta),$$
$$\frac{\partial}{\partial r}p(x,a-,\theta) = \zeta\frac{\partial}{\partial r}p(x,a+,\theta), \tag{4}$$

where

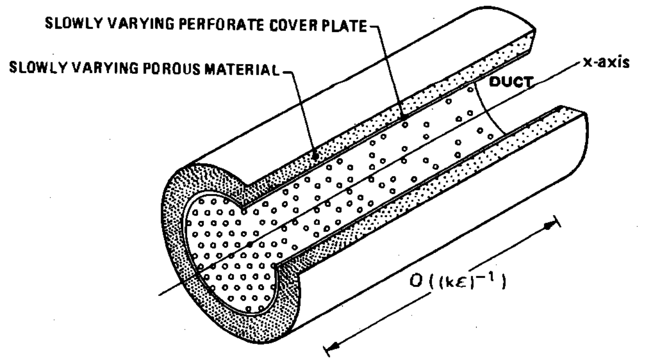

FIG. 2. Configuration of slowly varying liner.

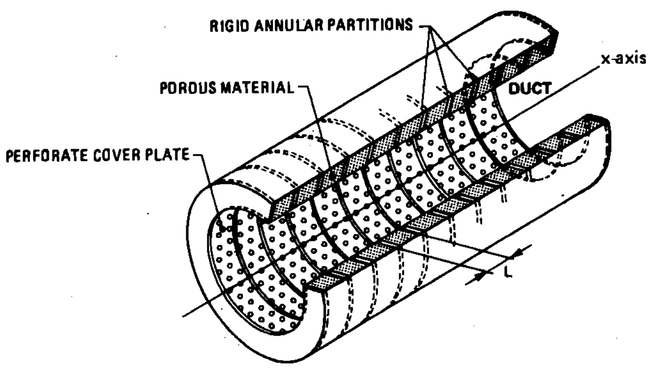

FIG. 3. Liner configuration with annular partitions.

$$\zeta = \rho_0/(\rho_e - i\sigma/\omega).$$

Note that the perforate plate is acoustically open if $Z = 0$. From this basis we will discuss in the following sections three configurations:

(1) The porous material is homogeneous, without segmented structure or partitioning (Fig. 1). This is the classical situation, and we will give here, for reference, a review of the established results,[3] appropriately generalized, and including a perforate cover plate.

(2) The properties of the porous material vary slowly in the axial direction (Fig. 2). To quantify this we introduce a small parameter ϵ (a typical axial acoustic wavelength/a typical length for order one variations of the material), and assume $\Omega, \sigma, \rho_e, c_e, \zeta$, and Z to be explicitly a function of ϵx.

(3) The (homogeneous) porous material is embedded in a segmented rigid structure of annular shape at $a \leqslant r \leqslant b$, $x = lL$, where $l = ..., -2, -1, 0, 1, 2, ...$ (Fig. 3). This axial partitioning is represented in the model as a boundary condition of vanishing normal velocity, implying

$$\frac{\partial}{\partial x} p(lL,r,\theta) = 0, \quad a \leqslant r \leqslant b, \quad l = ..., -1, 0, 1, ...$$

The problem considered is that of small L, relative to the acoustic wavelength.

II. HOMOGENEOUS POROUS MATERIAL

In this section we will briefly review the solution in the form of a modal expansion of the sound field, to obtain a convenient reference for the next section on slowly varying material. Combining Eqs. (1) gives the Helmholtz equation for p

$$(\nabla^2 + k^2)p = 0, \quad 0 \leqslant r \leqslant a, \tag{5}$$

where $k = \omega/c_0$; similarly Eqs. (2) yield

$$(\nabla^2 + \mu^2)p = 0, \quad a \leqslant r \leqslant b, \tag{6}$$

where $\mu = (\omega/c_e)(\Omega/\zeta)^{1/2}$, with the sign of the complex square root chosen such that $\mathrm{Re}\,\mu > 0$ and $\mathrm{Im}\,\mu < 0$, which makes the plane wave $\exp(-i\mu x)$ propagating in positive x direction.

The circular geometry allows a description in circumferential modes

$$p(x,r,\theta) = \sum_{m=-\infty}^{\infty} p_m(x,r) e^{-im\theta}. \tag{7}$$

Furthermore, each mode p_m can be built up from radial modes [this can be proved by constructing a solution for p_m via Fourier transformation in x; the resulting Fourier integral can be evaluated as an infinite sum of residues (modes)], resulting in

$$p_m(x,r) = \sum_{n=-\infty}^{\infty}{}' A_{mn} J_m(\alpha_{mn} r) \exp(-i\gamma_{mn} x), \quad (0 \leqslant r \leqslant a),$$

$$\begin{aligned} p_m(x,r) = \sum_{n=-\infty}^{\infty}{}' B_{mn} [&J_m(\beta_{mn} r) Y'_m(\beta_{mn} b) \\ &- Y_m(\beta_{mn} r) J'_m(\beta_{mn} b)] \\ &\times \exp(-i\gamma_{mn} x), \quad (a \leqslant r \leqslant b), \end{aligned} \tag{8}$$

where J_m and Y_m denote mth order Bessel functions[7]; a mode with $n > 0$ is right running, and with $n < 0$ left running, $n = 0$ being excluded. The wavenumbers γ, α, and β are related by dispersion relations, while combination of the interface coupling conditions yields an equation in α and β, which is the eigenvalue equation in γ, with an infinite set of solutions γ_{mn}:

$$\gamma^2 = k^2 - \alpha^2 = \mu^2 - \beta^2, \tag{9}$$

$$\begin{aligned} E(\gamma) = &\frac{J_m(\alpha a)}{\alpha a J'_m(\alpha a)} + \frac{Z}{i\omega\rho_0 a} - \frac{1}{\zeta\beta a} \\ &\times \frac{J_m(\beta a) Y'_m(\beta b) - Y_m(\beta a) J'_m(\beta b)}{J'_m(\beta a) Y'_m(\beta b) - Y'_m(\beta a) J'_m(\beta b)} = 0. \end{aligned}$$

Of course, each coupling condition alone provides an equation for the amplitude:

$$\begin{aligned} B_{mn} = A_{mn} &\frac{\alpha_{mn}}{\zeta\beta_{mn}} J'_m(\alpha_{mn} a) / [J'_m(\beta_{mn} a) Y'_m(\beta_{mn} b) \\ &- Y'_m(\beta_{mn} a) J'_m(\beta_{mn} b)] = A_{mn} D_{mn}. \end{aligned} \tag{10}$$

Substitution of the defining series of J_m and Y_m reveals that E is a meromorphic function (analytic everywhere except for isolated poles) of γ^2 (or α^2, or β^2). So we may assume

$$\gamma_{m,n} = -\gamma_{m,-n}.$$

From the physics of the problem it follows that the eigenvalues γ_{mn} are found in $\mathrm{Re}\,\gamma > 0$, $\mathrm{Im}\,\gamma < 0$, for $n > 0$, and in $\mathrm{Re}\,\gamma < 0$, $\mathrm{Im}\,\gamma > 0$, for $n < 0$. To obtain symmetry in the basis functions of left and right running modes, it is appropriate to define α and β with the following principal branch square roots:

$$\alpha = k(1 - \gamma/k)^{1/2}(1 + \gamma/k)^{1/2},$$

$$\beta = \mu(1 - \gamma/\mu)^{1/2}(1 + \gamma/\mu)^{1/2},$$

implying that $\alpha_{m,n} = \alpha_{m,-n}$, $\beta_{m,n} = \beta_{m,-n}$. Opposite values of $\alpha_{m,n}$ and $\beta_{m,n}$ may formally be included, but they are not related to other modes.

Since usually in applications the layer of porous material is relatively thin, it may be useful to note that for $d/a \to 0$ the eigenvalues α are given by

$$\alpha_{mn} a \simeq j'_{mn} \left[1 - \zeta \frac{d}{a} \left(1 + \frac{(\mu^2 - k^2)a^2}{j'^2_{mn} - m^2} \right) \right] \tag{11a}$$

for the first few ns, when βd is small, and

$$\alpha_{mn} a \simeq j'_{m,n} \left(1 + \frac{i\omega\rho_0 a}{(j'^2_{mn} - m^2) Z} \right), \tag{11b}$$

for higher n, when $\beta d > O(1)$ (and $Z \neq 0$). $j'_{m,n}$ denotes the nth zero of J'_m.

III. POROUS MATERIAL OF SLOWLY VARYING PROPERTIES

In this section we will derive a *multiple scale* solution (see, e.g., Nayfeh[5]) of the present problem with porous material of properties, slowly varying in the axial direction. To this end, we introduce a small, dimensionless parameter ϵ, such that the material parameters depend only on the combination ϵx: $\Omega = \Omega(\epsilon x)$, $\sigma = \sigma(\epsilon x)$, $\rho_e = \rho_e(\epsilon x)$, $c_e = c_e(\epsilon x)$, $Z = Z(\epsilon x)$, $\zeta = \zeta(\epsilon x)$, and $\mu = \mu(\epsilon x)$. This means that they show order one variations with order one variations of ϵx. Introduce for convenience the "slow variable" $X = \epsilon x$. The solution procedure is to decouple fast, acoustic variations in x from the slow variations in X. So locally, a wave will be describable as a modal sum like in the previous section, but on the slower scale amplitudes (and eigenvalues) will vary with X. The crux is then to find an equation in X alone, describing these variations.

Inside the duct $0 \leqslant r \leqslant a$ we have again Eq. (5); in the layer, however, combination of Eqs. (2) results in

$$\nabla^2 p + \epsilon(\zeta_X/\zeta)p_x + \mu^2 p = 0. \tag{12}$$

Assume a modelike waveform

$$p(x,r,\theta) = P(X,r)\exp\left(-\text{im}\,\theta - i\epsilon^{-1}\int^X \gamma(z)dz\right), \tag{13}$$

where P and γ are functions to be determined. Substitution in Eqs. (5) and (12) we obtain for P

$$\begin{aligned} &P_{rr} + (1/r)P_r - (m^2/r^2)P + (k^2 - \gamma^2)P \\ &\quad = i\epsilon(\gamma_X P + 2\gamma P_X) - \epsilon^2 P_{XX} \quad (0 \leqslant r \leqslant a), \end{aligned} \tag{14}$$

$$\begin{aligned} &P_{rr} + (1/r)P_r - (m^2/r^2)P + (\mu^2 - \gamma^2)P \\ &\quad = i\epsilon\{[\gamma_X + \gamma(\zeta_X/\zeta)]P + 2\gamma P_X\} \\ &\qquad - \epsilon^2[P_{XX} + (\zeta_X/\zeta)P_X] \quad (a \leqslant r \leqslant b). \end{aligned} \tag{15}$$

As an analogy the slow effects can be considered to act as a weak source. With respect to the left-hand sides, we observe a great similarity with the previous section, which indeed is a useful reference. We will use therefore the same symbols to define $\gamma^2(X) = k^2 - \alpha^2(X) = \mu^2(X) - \beta^2(X)$. Furthermore, to obtain a right-hand side equal in Eqs. (14) and (15), it is convenient to define $\zeta \equiv 1$ in $0 \leqslant r \leqslant a$. We proceed by assuming the expansion in ϵ

$$P(X,r,\epsilon) = P_0(X,r) + \epsilon P_1(X,r) + \cdots. \tag{16}$$

Substitution in Eqs. (14) and (15), and equating all like powers of ϵ, we obtain

$$\mathscr{L}(P_0) = 0, \tag{17}$$

$$\mathscr{L}(P_1) = i(\gamma_X + \gamma\,\zeta_X/\zeta)P_0 + 2i\gamma P_{0X} \tag{18}$$

(etc.) ($\zeta_X \equiv 0$ for $0 \leqslant r \leqslant a$), where the operator $\mathscr{L}$ is defined in $0 \leqslant r \leqslant a$ as

$$\mathscr{L} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \alpha^2(X) - \frac{m^2}{r^2}$$

and similarly, with β substituted for α, in $a \leqslant r \leqslant b$.

Boundary and interface conditions are to be applied to each term $P_0, P_1,\ldots,$ of Ps expansion. The solution of Eq. (17) is then [analogous to Eqs. (8)]

$$\begin{aligned} P_0(X,r) &= A(X)J_m(\alpha r) \quad (0 \leqslant r \leqslant a), \\ &= B(X)[J_m(\beta r)Y'_m(\beta b) \\ &\qquad - Y_m(\beta r)J'_m(\beta b)] \quad (a \leqslant r \leqslant b), \end{aligned} \tag{19}$$

where $\gamma(X)$ is a solution of $E[\gamma(X),X] = 0$, continuous in X [say, $\gamma = \gamma_{mn}(X)$], and $B(X) = A(X)D_{mn}(X)$ [see Eqs. (9)–(10)], thus X is acting as a parameter. Observe that the multiplicative "constant" (in r) $A(X)$ is still unknown; it has to be determined, more or less indirectly, from the conditions that P_0 must be such that Eq. (18) has a solution P_1. It appears that this *solvability condition* is just sufficient to determine A, and, moreover, it is possible to settle the equation for A without explicitly solving Eq. (18). This is established by multiplying left-and right-hand side of Eq. (18) with $\zeta r P_0$, and integrate to r from 0 to b: integration by parts and application of the boundary and interface conditions then yields

$$\int_0^a + \int_a^b \zeta r P_0 \mathscr{L}(P_1)\,dr = 0,$$

and so

$$\frac{d}{dX}\left(\gamma\int_0^a rP_0^2\,dr + \gamma\zeta\int_a^b rP_0^2\,dr\right) = 0.$$

Fortunately, these remaining integrals can be evaluated explicitly in analytic form (using some properties of Bessel functions[7]), to give

$$\begin{aligned} 0 &= \frac{d}{dX}A^2\gamma\left[\left(1 - \frac{m^2}{\alpha^2 a^2}\right)J_m(\alpha a)^2\right. \\ &\quad + \left(1 - \frac{\alpha^2}{\beta^2\zeta}\right)J'_m(\alpha a)^2 - \zeta\left(1 - \frac{m^2}{\beta^2 a^2}\right) \\ &\quad \left.\times\left(J_m(\alpha a) + \frac{\alpha Z}{i\omega\rho_0}J'_m(\alpha a)\right)^2 + \frac{4\zeta D^2}{\pi^2\beta^2 a^2}\left(1 - \frac{m^2}{\beta^2 b^2}\right)\right] \\ &= \frac{d}{dX}A(X)^2 f(X). \end{aligned} \tag{20}$$

Hence,

$$A(X) = A(0)[f(0)/f(X)]^{1/2} \tag{21}$$

which completes the description of a slowly varying "mode" to leading order. Evidently, higher order corrections are in principle possible but the present one is of course the most significant.

IV. ANNULAR PARTITIONING WITH SMALL PITCH

In this section we will discuss the problem of a sound field scattered by a segmented rigid structure of annular partitions in which the porous material is embedded. The difficult general problem will not be solved, but rather the behavior for a small pitch will be investigated.

The relevant equations here are again (5) and (6), but now with the extra boundary condition

$$\frac{\partial}{\partial x}p(x = lL, r, \theta) = 0$$

for $a \leqslant r \leqslant b$ and $l = \ldots, -1,0,1,\ldots$ [and, to be complete, the (so-called), edge condition, usual in diffraction problems, of a square root behavior of p near the edges $x = lL$, $r = a$].

The geometry considered allows, as before, a circumferential expansion in θ

$$p(x,r,\theta) = \sum_{m=-\infty}^{\infty} p_m(x,r)e^{-im\theta}.$$

Inside the porous material, between the rigid partitions $x = lL$ and $x = (l+1)L$, we may write

$$p_m(x,r) = \sum_{n=0}^{\infty} B_{mnl} \cos\left(\frac{\pi n x}{L}\right)[J_m(\lambda_n r)Y'_m(\lambda_n b) - Y_m(\lambda_n r)J'_m(\lambda_n b)], \quad (22)$$

where $a<r<b$, and $\lambda_n = [\mu^2 - (\pi n/L)^2]^{1/2}$. Note that the edge condition requires a certain behavior of B_{mnl} for $n \to \infty$.

Via Fourier cosine series expansion in x the coupling conditions at $r = a$ enable us to relate the coefficients B_{mnl} to the Fourier coefficients of $p_m(x,a-)$ and $(\partial/\partial r)p_m(x,a-)$. Together with an expression of the duct field, this forms an infinite system of integral equations, which solution is apparently very difficult to find. Instead we will concentrate on the limit $\mu L \to 0$ (say, $|\mu L| < \pi$). Then the modes $n \neq 0$ in the liner are exponentially damped in r direction, and do not contribute very much to the field, except near the interface $r = a$. So for not too small d ($= b - a$) the duct field is mainly coupled to the only cut-on (m) mode $n = 0$, and we can restrict use to the equations

$$\int_{lL}^{(l+1)L} p_m(x,a-)dx = \int_{lL}^{(l+1)L} \frac{\partial}{\partial r} p_m(x,a-)dr\left(\frac{1}{\zeta\mu} \cdot \frac{J_m(\mu a)Y'_m(\mu b) - Y_m(\mu a)J'_m(\mu b)}{J'_m(\mu a)Y'_m(\mu b) - Y'_m(\mu a)J'_m(\mu b)} - \frac{Z}{i\omega\rho_0}\right) \quad (23)$$

$$= B_{m0l}L\left(J_m(\mu a)Y'_m(\mu b) - Y_m(\mu a)J'_m(\mu b) - \frac{Z\zeta\mu}{i\omega\rho_0}[J'_m(\mu a)Y'_m(\mu b) - Y'_m(\mu a)J'_m(\mu b)]\right). \quad (24)$$

The averaged field quantities of Eq. (23) do not approximate the field locally along $r = a$, since near the edges the diffracted waves vary rapidly. However, the region of influence of the edges is only of order L, so slightly away from the interface the duct field is axially approximately constant, and equal to the averaged value. Thus we may write Eq. (23) in the form of an (m-dependent) impedance for an m mode of the duct field. At $lL = x$ we have

$$\frac{p_m(x,a)}{v_m(x,a)} = Z - \frac{i\omega\rho_0}{\zeta\mu} \times \frac{J_m(\mu a)Y'_m(\mu b) - Y_m(\mu a)J'_m(\mu b)}{J'_m(\mu a)Y'_m(\mu b) - Y'_m(\mu a)J'_m(\mu b)}, \quad (25)$$

where v_m is the m component of the radial velocity.

Since a liner is usually designed as thin as possible, it may be useful to note that for $\mu d \to 0$ (but $d \gg L$) (25) takes the form

$$\frac{p_m(x,a)}{v_m(x,a)} \simeq \frac{-i\omega\rho_0}{\zeta\mu}\left(1 - \frac{m^2}{\mu^2 a^2}\right)^{-1}(\mu d)^{-1}. \quad (26)$$

Clearly, the liner is point reacting in the axial direction, but is really of the "bulk type" circumferentially. However, the equivalent impedance boundary condition makes any calculation method for cylindrical ducts with locally reacting lining applicable to ducts with the present type of lining. Moreover, since we *only* used the circumferential periodicity of the duct field, the present analysis is equally valid for ducts with (axial) mean flow and other similar variations of the mean medium.

V. CONCLUDING REMARKS

Some problems of sound propagation in cylindrical ducts with a lining of porous material have been discussed theoretically. Apart from a review of the modal solution for a homogeneous liner, analytical solutions for two configurations have been derived: for a liner whose properties vary slowly in the axial direction, and for a liner of porous material embedded in a rigid structure of annular partitions with a small pitch.

The problem of the varying liner has been solved by using the (so-called) multiple scale technique. The solution is built up from modelike solutions with slowly varying amplitude and axial wavenumber. Remarkably, the (leading order) equations could be integrated explictly, giving a solution practically in analytic form (only the eigenvalue equation has to be solved for each axial position). Although the practical possibilities have to be subject to further study, it is evident that the extra degree of freedom will enable a designer, in principle at least, to increase the damping capacities of a liner.

The problem of the liner with annular partitions of small pitch has been solved by realizing that in a liner section only the lowest order of a circumferential (m) mode is able to propagate. This allows, per m, a decoupling of the liner field and duct field, to obtain effectively a quasilocally reacting liner whose properties are described by an m-dependent impedance. So for an arbitrary sound field with several m-components the liner has properties inbetween locally and nonlocally reacting. At the same time any calculation procedure for sound propagation in ducts which employs a circumferential expansion of the sound field, and designed for ducts with locally reacting liners is immediately applicable to liners of the present type.

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