

Cut-on, cut-off transition of sound in slowly varying flow ducts

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Abstract

The cut-on/cut-off transition of a slowly varying acoustic mode in a slowly varying hard-walled duct with irrotational isentropic mean flow is studied. The usual turning point behaviour was found.

1 Introduction

The exact multiple scale solution for sound propagation in a slowly varying lined flow duct Rienstra (1999) describes in the form of a WKB approximation the way a cylindrical duct mode propagates through a lined duct with mean flow, for example a turbofan aircraft engine.

Apart from direct aeronautical engine applications (Rienstra & Eversman 2001), one of the opportunities this solution offers is a better understanding of the cut-on/cut-off transition of a mode in a hard-wall flow duct, also known as a turning point problem. This was done for the no-flow case by Nayfeh & Telionis (1973). Here, we will extend this to include the effect of flow.

2 Physical model

Consider inside the cylinder $r = R(X)$, $X = \varepsilon x$, an isentropic irrotational flow, described by velocity $\tilde{\mathbf{v}}$, potential $\tilde{\phi}$, density $\tilde{\rho}$, pressure \tilde{p} and soundspeed \tilde{c} . We split it up in a stationary part and acoustic harmonic perturbations

$$\tilde{\mathbf{v}} = \mathbf{V} + \mathbf{v} e^{i\omega t}, \quad \tilde{\phi} = \Phi + \phi e^{i\omega t}, \quad \tilde{\rho} = D + \rho e^{i\omega t}, \quad \tilde{p} = P + p e^{i\omega t}, \quad \tilde{c} = C + c e^{i\omega t}.$$

This yields for mean flow field (in dimensionless form)

$$\nabla \cdot (D\mathbf{V}) = 0, \quad \frac{1}{2}|\mathbf{V}|^2 + \frac{C^2}{\gamma - 1} = E, \quad C^2 = \frac{\gamma P}{D} = D^{\gamma-1}. \quad (1)$$

γ is the ratio of specific heats, while E is the Bernoulli constant. For the acoustic field we have

$$i\omega\rho + \nabla \cdot (D\nabla\phi + \rho\mathbf{V}) = 0, \quad i\omega\phi + \mathbf{V} \cdot \nabla\phi + \frac{p}{D} = 0, \quad p = C^2\rho. \quad (2)$$

(The sound speed perturbations play no role, of course).

As we are interested in the cut-on/cut-off transition, which in general does not occur for modes in a soft-walled duct, we consider only a solid duct wall, so the normal velocity vanishes at the wall. See Rienstra (1999) for the more general case of a lined wall.

The mean flow is assumed to be free of swirl, irrotational and nearly uniform, while duct entrance effects are absent. This implies that the mean flow field is essentially a function of X , rather than x , and to leading order given by

$$\mathbf{V}(X, r; \varepsilon) = U(X)\mathbf{e}_x + \varepsilon V_1(X, r)\mathbf{e}_y + O(\varepsilon^2)$$

Whenever possible, we ignore in the asymptotic expansions the implied indices. The mean flow pressure, density and sound speed are to leading order also functions of only X (Rienstra 1999).

After elimination of p and ρ , the acoustic field is described by

$$\nabla \cdot (D\nabla\phi) - D\left(\mathrm{i}\omega + \mathbf{V} \cdot \nabla\right) \left[\frac{1}{C^2} \left(\mathrm{i}\omega + \mathbf{V} \cdot \nabla\right) \phi \right] = 0, \quad (3)$$

3 The multiple scales solution

For a uniform flow in a straight duct, the acoustic field is known to be described by circumferential-radial modes. In accordance with WKB theory, the modal spectrum of waves in a slowly varying medium varies along with the slow variations, with only very little intermodal interference if the eigenvalues are well enough separated. Therefore, we introduce for the slowly varying case the WKB-Ansatz

$$\phi(x, r, \theta; \varepsilon) = A(X, r; \varepsilon) \exp\left(-\mathrm{i}m\theta - \mathrm{i}\varepsilon^{-1} \int^X \mu(\xi; \varepsilon) \mathrm{d}\xi\right) \quad (4)$$

where ϕ will be any slowly varying ‘mode’. Following (Rienstra 1999), we obtain after expanding $A(X, r; \varepsilon) = A(X, r) + O(\varepsilon)$, $\mu(X; \varepsilon) = \mu(X) + O(\varepsilon^2)$ from the leading order equation the solution

$$A(X, r) = N(X) J_m(\alpha(X)r) \quad (5)$$

where J_m is the m -th order Bessel function of the first kind (Abramowitz & Stegun 1964). The boundary condition of vanishing normal velocity yields to leading order the following equation for ‘eigenvalue’ $\alpha(X)$

$$J'_m(\alpha(X)R(X)) = 0.$$

With j'_{mn} the n -th zero of J'_m , we have $\alpha = j'_{mn}/R$. α and μ are related by the dispersion relation

$$C^2(\alpha^2 + \mu^2) = (\omega - \mu U)^2.$$

It is convenient to introduce the reduced axial and radial wave numbers σ and γ , which are really μ and α scaled by ω and without the pure convection effects, such that

$$\sigma = \sqrt{1 - \gamma^2}, \quad \text{where } \gamma = \frac{j'_{mn}}{\omega} \frac{\sqrt{C^2 - U^2}}{R}, \quad \text{and } \mu = \omega \frac{C\sigma - U}{C^2 - U^2}.$$

We define the square root such that $\mathrm{Re}(\sigma) \geq 0$, and $\mathrm{Im}(\sigma) \leq 0$. For the present hard-wall geometry γ is only real. When $\gamma < 1$, σ is real positive and the mode is called propagating or cut-on. When $\gamma > 1$, σ is negative imaginary and the mode is called evanescent or cut-off.

By various manipulations applied to the higher order equations (see (Rienstra 1999) for details), we obtain the following expression for amplitude $N(X)$

$$N = Q \left(\frac{C}{R^2 D \sigma} \right)^{1/2}, \quad (6)$$

where Q is a constant.

4 The turning point problem

An important property of expression (6) for N is that it becomes invalid when $\sigma = 0$. So when the medium and diameter vary in such a way that at some point $X = X_0$ wave number σ vanishes, the present solution breaks down. In a small interval around X_0 the mode does *not* vary slowly and locally a different approximation is necessary. In the terminology of Matched Asymptotic Expansions (Holmes 1995), this is a boundary layer in variable X .

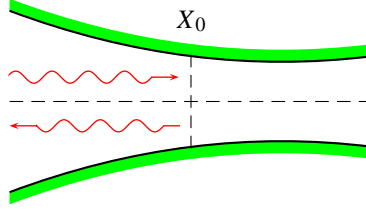


Figure 1: Turning point X_0 , where a mode changes from cut-on to cut-off.

When σ^2 changes sign, and σ changes from real into imaginary, the mode changes from cut-on to cut-off. If X_0 is isolated, such that there are no interfering neighbouring points of vanishing σ , no power is transmitted beyond X_0 , and the wave has to reflect at X_0 . The incident propagating mode is split up into a cut-on reflected mode and a cut-off transmitted mode. Therefore, a point where wave number σ vanishes is called a “turning point”.

Assume at $X = X_0$ a transition from cut-on to cut-off, so

$$\sigma_0 = 0, \quad \frac{d}{dX} \sigma_0^2 < 0, \quad \gamma_0 = 1, \quad \gamma'_0 > 0, \quad \frac{C_0 C'_0 - U_0 U'_0}{C_0^2 - U_0^2} - \frac{R'_0}{R_0} > 0.$$

where subscript “0” indicates evaluation at $X = X_0$.

Consider an incident, reflected and transmitted wave of the type found above. So in $X < X_0$, where σ is real positive, we have the incident and reflected waves

$$\phi = \frac{n(X)}{\sqrt{\sigma(X)}} J_m(\alpha r) e^{\frac{i}{\varepsilon} \int_{X_0}^X \frac{\omega U}{C^2 - U^2} dX'} \left[e^{-\frac{i}{\varepsilon} \int_{X_0}^X \frac{\omega C \sigma}{C^2 - U^2} dX'} + \mathcal{R} e^{\frac{i}{\varepsilon} \int_{X_0}^X \frac{\omega C \sigma}{C^2 - U^2} dX'} \right] \quad (7)$$

with reflection coefficient \mathcal{R} to be determined and

$$n(X) = Q \left(\frac{C}{D R^2} \right)^{1/2}.$$

In $X > X_0$, where σ is imaginary negative, we have the transmitted wave

$$\phi = \mathcal{T} \frac{n(X)}{\sqrt{\sigma(X)}} J_m(\alpha r) e^{\frac{i}{\varepsilon} \int_{X_0}^X \frac{\omega U}{C^2 - U^2} dX'} e^{-\frac{1}{\varepsilon} \int_{X_0}^X \frac{\omega C |\sigma|}{C^2 - U^2} dX'} \quad (8)$$

with transmission coefficient \mathcal{T} to be determined, while $\sqrt{\sigma} = e^{-\frac{1}{4}\pi i} \sqrt{|\sigma|}$ will be taken.

This set of approximate solutions of equation (3), valid outside the turning point region, constitute the outer solution. Inside the turning point region this approximation breaks down. The approximation is invalid here, because neglected terms of equation (3) are now dominant, and another approximate equation is to be used. This will give us the inner or boundary layer solution. To determine the unknown constants (here: \mathcal{R} and \mathcal{T}), inner and outer solution are asymptotically matched.

For the matching it is necessary to determine the asymptotic behaviour of the outer solution in the limit $X \rightarrow X_0$, and the boundary layer thickness (*i.e.* the appropriate local coordinate).

From the limiting behaviour of the outer solution in the turning point region (see below), we can estimate the order of magnitude of the solution. From a balance of terms in the differential equation (3) it transpires that the turning point boundary layer is of thickness $X - X_0 = O(\varepsilon^{2/3})$, leading to a boundary layer variable ξ given by

$$X = X_0 + \varepsilon^{2/3} \lambda^{-1} \xi$$

where λ is introduced for notational convenience later, and is given by

$$\lambda^3 = \frac{2\omega^2 C_0^2}{(C_0^2 - U_0^2)^2} \left(\frac{C_0 C_0' - U_0 U_0'}{C_0^2 - U_0^2} - \frac{R_0'}{R_0} \right).$$

Since for $\varepsilon \rightarrow 0$

$$\sigma^2(X) = \sigma^2(X_0 + \varepsilon^{2/3} \xi) = -2\varepsilon^{2/3} \left(\frac{C_0 C_0' - U_0 U_0'}{C_0^2 - U_0^2} - \frac{R_0'}{R_0} \right) \lambda^{-1} \xi + O(\varepsilon^{4/3} \xi^2),$$

we have

$$\frac{1}{\varepsilon} \int_{X_0}^X \frac{\omega C \sigma}{C^2 - U^2} dX' = \begin{cases} -\frac{2}{3} (-\xi)^{3/2} = -\zeta, & \text{if } \xi < 0 \\ -i \frac{2}{3} \xi^{3/2} = -i\zeta, & \text{if } \xi > 0 \end{cases}$$

where we introduced $\zeta = \frac{2}{3} |\xi|^{3/2}$. The limiting behaviour for $X \uparrow X_0$ is now given by

$$\phi \simeq \frac{n_0}{\varepsilon^{1/6} (-\xi)^{1/4}} \left(\frac{\omega C_0}{\lambda (C_0^2 - U_0^2)} \right)^{1/2} J_m(\alpha_0 r) e^{-im\theta} \left(e^{i\zeta} + \mathcal{R} e^{-i\zeta} \right), \quad (9)$$

while it is for $X \downarrow X_0$ given by

$$\phi \simeq \mathcal{T} \frac{n_0}{\varepsilon^{1/6} \xi^{1/4}} \left(\frac{\omega C_0}{\lambda (C_0^2 - U_0^2)} \right)^{1/2} e^{\frac{1}{4}\pi i} J_m(\alpha_0 r) e^{-im\theta} e^{-\zeta}. \quad (10)$$

Since the boundary layer is relatively thin, also compared to the radial coordinate, the behaviour of the incident mode remains rather unaffected in radial direction, and we can assume in the turning point region

$$\phi(x, r, \theta) = \psi(\xi) J_m(\alpha(X)r) e^{-im\theta} e^{\frac{i}{\varepsilon} \int_{X_0}^X \frac{\omega U}{C^2 - U^2} dX'}.$$

Substitution in equation (3), and using the definition of the Bessel equation, we arrive at

$$\left(1 - \frac{U_0^2}{C_0^2} \right) \varepsilon^{2/3} \lambda^2 J(\alpha_0 r) (\psi'' - \xi \psi) = O(\varepsilon).$$

So to leading order we have Airy's equation

$$\frac{d^2\psi}{d\xi^2} - \xi\psi = 0.$$

This has the general solution (figure 2)

$$\psi(\xi) = a\text{Ai}(\xi) + b\text{Bi}(\xi),$$

where a and b , parallel with \mathcal{R} and \mathcal{T} , are to be determined from matching. Using the asymptotic expressions (13,14) for Airy functions, we find that for ξ large with $1 \ll \xi \ll \varepsilon^{-2/3}$, equation

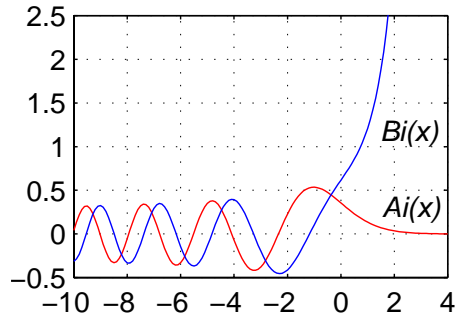


Figure 2: Airy functions

(10) matches the inner solution if

$$\frac{a}{2\sqrt{\pi}\xi^{1/4}} e^{-\zeta} + \frac{b}{\sqrt{\pi}\xi^{1/4}} e^{\zeta} \sim \mathcal{T} \frac{n_0}{\varepsilon^{1/6}\xi^{1/4}} e^{\frac{1}{4}\pi i} \left(\frac{\omega C_0}{\lambda(C_0^2 - U_0^2)} \right)^{1/2} e^{-\zeta}.$$

Since $e^{\zeta} \rightarrow \infty$, we can only have $b = 0$, and thus

$$a = \frac{2n_0\sqrt{\pi}}{\varepsilon^{1/6}} \left(\frac{\omega C_0}{\lambda(C_0^2 - U_0^2)} \right)^{1/2} e^{\frac{1}{4}\pi i} \mathcal{T}$$

If $-\xi$ is large with $1 \ll -\xi \ll \varepsilon^{-2/3}$ we use the asymptotic expression (13), and find that equation (9) matches the inner solution if

$$a \frac{\cos(\zeta - \frac{1}{4}\pi)}{\sqrt{\pi}(-\xi)^{1/4}} \sim \frac{n_0}{\varepsilon^{1/6}(-\xi)^{1/4}} \left(\frac{\omega C_0}{\lambda(C_0^2 - U_0^2)} \right)^{1/2} (e^{i\zeta} + \mathcal{R} e^{-i\zeta}),$$

or

$$\mathcal{T} e^{i\zeta} + \mathcal{T} i e^{-i\zeta} \sim e^{i\zeta} + \mathcal{R} e^{-i\zeta}.$$

So, finally, we have

$$T = 1, \quad R = i. \quad (11)$$

The amplitudes of these reflection and transmission coefficients could of course be guessed by conservation of energy arguments. This is not the case with the phase. It appears that the wave reflects with a phase change of $\frac{1}{2}\pi$, while the transmission is without phase change.

5 Conclusions

A loose end in the exact solution of the multiple scales problem of sound transmission in a slowly varying flow duct was the transition through cut-on/cut-off. By this publication we hope to have amended this omission.

Appendix

Related to Bessel functions of order $\frac{1}{3}$ are the Airy functions Ai and Bi, solution of

$$y'' - xy = 0, \quad (12)$$

with the following asymptotic behaviour (introduce $\zeta = \frac{2}{3}|x|^{3/2}$)

$$\text{Ai}(x) \simeq \begin{cases} \frac{\cos(\zeta - \frac{1}{4}\pi)}{\sqrt{\pi}|x|^{1/4}} & (x \rightarrow -\infty), \\ \frac{e^{-\zeta}}{2\sqrt{\pi}x^{1/4}} & (x \rightarrow \infty), \end{cases} \quad (13)$$

$$\text{Bi}(x) \simeq \begin{cases} \frac{\cos(\zeta + \frac{1}{4}\pi)}{\sqrt{\pi}|x|^{1/4}} & (x \rightarrow -\infty), \\ \frac{e^{\zeta}}{\sqrt{\pi}x^{1/4}} & (x \rightarrow \infty). \end{cases} \quad (14)$$

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