The dynamics of a suspended pipeline, in the limit of vanishing stiffness[1]

by

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Abstract

This paper discusses a model for the linearized dynamics of offshore pipelines during laying in one plane. The linearized, two-dimensional equations of pipeline dynamics are solved using an approximate analytical and a numerical solution. It is shown that the solutions compare well and the range of validity of the approximation is considered.

1 Introduction

Exploration and production of oil and natural gas in deep water offshore requires the presence of pipelines for transport of the products ([1],[2],[3],[4],[5], [8],[12]). These pipelines are laid onto the seafloor by a pipelay ship. While producing more pipeline, the ship moves steadily away from the laid part of the pipeline, and the new part is suspended from the laybarge down to the sea floor. When the pipeline is suspended, it is bent under its own weight, and there is a serious risk of damage by pipe buckling due to too high bending stresses. To avoid this buckling, the pipeline is tensioned by a horizontal pulling force applied by the ship. The amount of tension that is sufficient is a major design problem for the pipelay engineer.

When the sea is calm, this pipelay process is practically time-independent. However, when the ship starts to move up and down on the waves, the

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forces on the pipeline will become time-dependent, and it becomes unclear whether the bending stresses remain low enough to be safe.

When the pipeline is relatively stiff (like in shallow water) the resonance frequencies of the suspended pipeline will be much higher than the typical water wave frequency, and the geometry and corresponding stresses will simply follow the heaving motion of the ship. The more interesting configuration is therefore the very slender pipe of very low stiffness. This is what we will consider here.

Basically, the pipe motion can be theoretically analysed by using the theory of slender beams with large deflection. The contributions of this paper is the derivation and analysis of the simplest model that contains the essentials of the dynamics of a pipeline of vanishing stiffness. Mathematically, a model of the dynamics of pipelines is described by a differential equation for the position vector. The displacement of the pipeline is assumed to be small around a mean position, which is the static configuration, so that we can linearize the equations.

A fully numerical solution method will be employed here, as well as an analytical approximation utilizing a large tension compared to weight. It is shown that in a relatively large parameter range the solutions compare well.

2 Formulation

The analysis of pipe motion here will be restricted to the case of motion in one plane. Torsion and friction with surrounding water are neglected. The two dimensional equations of motion are written as a differential equation for the position vector.

The pipe is described by the position vector $\mathbf{r}(s,t) = (x(s,t), y(s,t), 0)$ of the pipe axis as function of curve length s and time t, with natural local coordinate s such that $|\mathbf{r}'| = 1$, where $\{\}' = \frac{\partial}{\partial s}\{\}$ and $\{\dot{}\} = \frac{\partial}{\partial t}\{\}$ (see for example [10]). Introduce the tangential unit vector $\mathbf{t} = \mathbf{r}'$, the principal normal unit vector $\mathbf{n}$, and binormal unit vector $\mathbf{b}$, such that $\mathbf{b} = \mathbf{t} \times \mathbf{n}$, $\mathbf{n} = \mathbf{b} \times \mathbf{t}$, $\mathbf{t} = \mathbf{n} \times \mathbf{b}$. The curvature vector is $\mathbf{k} = \mathbf{t}' = \mathbf{r}''$, with curvature

$|\kappa| = |\mathbf{k}|$ defined such that $\mathbf{k} = \kappa\mathbf{n}$. The torsion or second curvature vector is $\mathbf{b}' = -\tau\mathbf{n}$, with torsion τ. Note that $\mathbf{n}' = -\kappa\mathbf{t} + \tau\mathbf{b}$. The angle between horizon and tangent $\mathbf{t}(s,t)$ is $\phi(s,t)$ (the pipe is configured in the z=0 -plane only), so we have

$$x(s,t) = x(0,t) + \int_0^s \cos\phi(s,t)\,\mathrm{d}s',$$
$$y(s,t) = y(0,t) + \int_0^s \sin\phi(s,t)\,\mathrm{d}s'.$$

Note that $\kappa^2 = |\mathbf{r}''|^2 = (\phi')^2$.

Introduce a pipe element of length ds, loaded by an external line load $\mathbf{q}$ and internal forces $\mathbf{F}$ and moments $\mathbf{M}$ at the both ends. The basic equations are derived from the equilibrium of the dynamic forces, equilibrium of the moments, and from the constitutive equations as follows (see ref.[9]).

For a beam there is a moment around $\mathbf{b}$ (bending) and around $\mathbf{t}$ (torsion) so $\mathbf{M} = M_B\mathbf{b} + M_T\mathbf{t}$. Torsion will be assumed to be zero, and M_B is given by the following constitutive equation of bending

$$M_B = EI\kappa \tag{1}$$

where bending stiffness EI is the product of Young's modulus E and the moment of inertia I. Since the force $\mathbf{F}$ is the only cause of the deformation, $\mathbf{F}$ lies in the plane of tangent and principal normal, so $\mathbf{F} = T\mathbf{t} + S\mathbf{n}$, where T is called the normal force and S the shearing force. The dynamic force equilibrium $\mathrm{d}\mathbf{F} + \mathbf{q}\,\mathrm{d}s = m_0\ddot{\mathbf{r}}\,\mathrm{d}s$ (where m_0 is the mass per unit length) yields

$$\mathbf{F}' + \mathbf{q} = m_0\ddot{\mathbf{r}} \tag{2}$$

and the moment equilibrium $\mathrm{d}\mathbf{M} + \mathrm{d}\mathbf{r} \times \mathbf{F} = 0$ becomes

$$\mathbf{M}' + \mathbf{t} \times \mathbf{F} = 0. \tag{3}$$

Furthermore, from the following vector identity applied to (3)

$$\mathbf{t} \times (\mathbf{M}' + \mathbf{t} \times \mathbf{F}) = \mathbf{t} \times \mathbf{M}' + \mathbf{t} \times (\mathbf{t} \times \mathbf{F}) =$$
$$\mathbf{t} \times \mathbf{M}' + \mathbf{t}(\mathbf{t} \cdot \mathbf{F}) - \mathbf{F}(\mathbf{t} \cdot \mathbf{t}) = \mathbf{t} \times \mathbf{M}' + T\mathbf{t} - \mathbf{F} = 0$$

we find with (2)

$$(\mathbf{t} \times \mathbf{M}')' + (T\mathbf{t})' + \mathbf{q} = m_0 \ddot{\mathbf{r}}. \tag{4}$$

With (1) and $M_T = 0$, we have

$$\begin{aligned}
\mathbf{t} \times \mathbf{M}' &= (\mathbf{t} \times \mathbf{M})' - \mathbf{t}' \times \mathbf{M} \\
&= \Big(\mathbf{t} \times (M_B \mathbf{b} + M_T \mathbf{t})\Big)' - \kappa \mathbf{n} \times (M_B \mathbf{b} + M_T \mathbf{t}) \\
&= -(M_B \mathbf{n})' - M_B \kappa \mathbf{t} + M_T \kappa \mathbf{b} \\
&= -EI(\kappa \mathbf{n})' - EI\kappa^2 \mathbf{t} \\
&= -EI\mathbf{k}' - EI\kappa^2 \mathbf{t}.
\end{aligned}$$

Substituting this equation into (4), yields the following differential equation for the radius vector

$$\Big(-EI(\mathbf{r}''' + \kappa^2 \mathbf{r}') + T\mathbf{r}'\Big)' + \mathbf{q} = m_0 \ddot{\mathbf{r}}. \tag{5}$$

In this problem the external forces acting on the pipe are assumed to consists only of the weight of the pipe, corrected for buoyancy. So here $\mathbf{q} = (0, -Q, 0)$, where $Q = m_0 g - B$ is the submerged weight per pipe length, g is the acceleration of gravity, and B is the buoyancy per pipe length. Written out in x and y coordinates, equation (5) becomes

$$(EI\phi'' \sin\phi + T\cos\phi)' = m_0 \ddot{x} \tag{6a}$$

$$(-EI\phi'' \cos\phi + T\sin\phi)' - Q = m_0 \ddot{y} \tag{6b}$$

When we scale lengths on the stationary free length L_0, forces on the stationary horizontal force H, and time on the applied frequency ω, we obtain in dimensionless variables s,T,x,y the following expressions that may provide us with information on the relative importance of the various terms:

$$\left(\frac{EI}{HL_0^2}\phi'' \sin\phi + T\cos\phi\right)' = \frac{m_0 \omega^2 L_0^2}{H}\ddot{x}$$

$$\left(-\frac{EI}{HL_0^2}\phi'' \cos\phi + T\sin\phi\right)' - \frac{QL_0}{H} = \frac{m_0 \omega^2 L_0^2}{H}\ddot{y}$$

The problem we are interested in is one with a horizontal pulling force H that is usually just enough to carry the pipeline, so $QL_0/H = O(1)$; a dynamical behaviour that is neither low- nor high-frequent, so $\omega L_0\sqrt{m_0/H} = O(1)$; and a small relative bending stifness, so $EI/HL_0^2 \to 0$. Except for boundary layer regimes near the ends (which is a separate problem that will not be considered here; see for the corresponding stationary case [4],[6],[8],[11],[12],[14]) equation (5) therefore simplifies to

$$(T\mathbf{r}')' + \mathbf{q} = m_0\ddot{\mathbf{r}}, \tag{7}$$

or in x and y coordinates (the reduced equations (6a) and (6b))

$$(T\cos\phi)' = m_0\ddot{x}, \tag{8a}$$
$$(T\sin\phi)' - Q = m_0\ddot{y} \tag{8b}$$

which are the dynamic equations for the 2-D problem to be treated here.

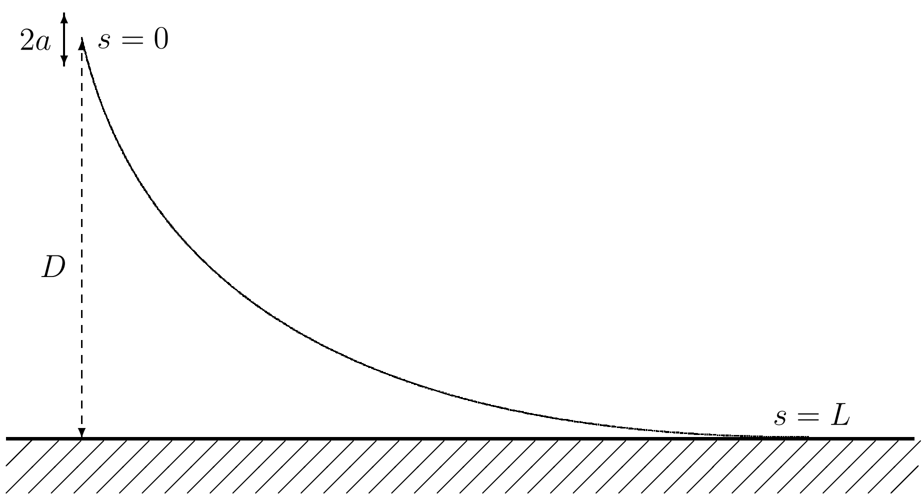

Figure 1: The configuration of the suspended pipeline

To minimize the role of stiffness we will consider here the J-lay method (see Figure 1) where the pipe is layed without a stinger. Then at the upper end the pipe is assumed to be hinged, i.e. the bending moment vanishes. Similarly, the bending moment vanishes at the freely laying touch-down point at $s = L$. Suitable boundary conditions are given by

$$\phi(L,t) = 0, \tag{9a}$$
$$y(L,t) = 0, \tag{9b}$$
$$x(0,t) = 0, \tag{9c}$$
$$y(0,t) = D + a\cos(\omega t), \tag{9d}$$
$$T(0,t)\cos\phi(0,t) = H, \tag{9e}$$

where D is the depth of water, L=$L(t)$ is the length of pipe in dynamic condition, a and ω are the amplitude and frequency of the heaving motion, and H is the constant horizontal pulling force of the laybarge. With (8a) and (8b), we get the static equations

$$(T_0\cos\phi_0)' = 0, \tag{10a}$$
$$(T_0\sin\phi_0)' = Q \tag{10b}$$

with boundary condition $\phi_0(L_0) = 0$ and $y_0(L_0) = 0$, where index $_0$ denotes the respective variable in static condition. Note that the direction of the stationary component of $\mathbf{F}$ is parallel to the pipe.

2.1 Linearized Dynamics

The linearized dynamics equations are obtained from (7) by assuming that the motion consists of small pertubations around the mean static configuration. Then the dynamic pipeline behaviour can be obtained analytically for small μ, or numerically.
Assume

$$T = T_0 + \tau,$$
$$\phi = \phi_0 + \psi,$$
$$L = L_0 + \ell$$

with harmonic pertubation in motion τ, ψ, $\ell \sim \cos\omega t$. With this and (9d) we expand

$$x(s,t) = \int_0^s \cos\phi_0 \,\mathrm{d}s' - \int_0^s \psi \sin\phi_0 \,\mathrm{d}s' + \cdots,$$
$$y(s,t) = D + a\cos\omega t + \int_0^s \sin\phi_0 \,\mathrm{d}s' + \int_0^s \psi \cos\phi_0 \,\mathrm{d}s' + \cdots,$$

we may split off the perturbed part, and by writing

$$\tau = \tilde{\tau}\cos\omega t,$$
$$\psi = \tilde{\psi}\cos\omega t,$$

the above equations (8a) and (8b) become after differentiation to s

$$\Big[\tilde{\tau}\cos\phi_0 - \tilde{\psi}T_0\sin\phi_0\Big]'' - m_0\omega^2\tilde{\psi}\sin\phi_0 = 0, \tag{11a}$$
$$\Big[\tilde{\tau}\sin\phi_0 + \tilde{\psi}T_0\cos\phi_0\Big]'' + m_0\omega^2\tilde{\psi}\cos\phi_0 = 0. \tag{11b}$$

Since we differentiated another time to get rid of x and y in favour of $\tilde{\psi}$, we need extra boundary conditions as follows

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big[\tilde{\tau}\cos\phi_0 - \tilde{\psi}T_0\sin\phi_0\Big] = 0 \qquad \text{at} \quad s = 0, \tag{12a}$$
$$\frac{\mathrm{d}}{\mathrm{d}s}\Big[\tilde{\tau}\sin\phi_0 + \tilde{\psi}T_0\cos\phi_0\Big] = -m_0\omega^2 a \qquad \text{at} \quad s = 0. \tag{12b}$$

From equations (10a) and (10b), we get

$$T_0\cos(\phi_0) = H,$$
$$T_0\sin(\phi_0) = Q(s - L_0).$$

Therefore, the static solutions are

$$\phi_0 = \arctan\frac{Q}{H}(s - L_0), \tag{13}$$
$$T_0 = \sqrt{H^2 + Q^2(s - L_0)^2}. \tag{14}$$

Considering the condition

$$y(L_0) = D + \int_0^{L_0} \sin\phi_0 \,\mathrm{d}s' = 0$$

we find that

$$L_0 = \sqrt{D^2 + 2D\frac{H}{Q}}. \tag{15}$$

Using these equations we may write the boundary conditions as follows. From (9e), we get

$$T_0 \cos\phi_0 + \tau\cos\phi_0 - \psi T_0 \sin\phi_0 = H \qquad \text{at } s = 0,$$

so

$$QL_0\,\tilde{\psi} + \frac{H}{\sqrt{H^2 + Q^2L_0^2}}\,\tilde{\tau} = 0 \qquad \text{at } s = 0,$$

and equation (12a) can be written as

$$\tilde{\tau}'\cos\phi_0 - \tilde{\tau}\phi_0'\sin\phi_0 + QL_0\,\tilde{\psi}' - Q\,\tilde{\psi} = 0 \qquad \text{at } s = 0,$$

then we get

$$QL_0\Big[L_0\,\tilde{\psi}' - \tilde{\psi}\Big] + \frac{H}{\sqrt{H^2 + Q^2L_0^2}}\Big[L_0\,\tilde{\tau}' + \frac{Q^2L_0^2}{H^2 + Q^2L_0^2}\,\tilde{\tau}\Big] = 0 \quad \text{at } s = 0.$$

From (12b) we have

$$\tilde{\tau}'\sin\phi_0 + \tilde{\tau}\phi_0'\cos\phi_0 + H\,\tilde{\psi}' = -am_0\omega^2 \quad \text{at } s = 0,$$

or

$$H\sqrt{H^2 + Q^2L_0^2}\,\tilde{\psi}' - QL_0\,\tilde{\tau}' + \frac{QH^2}{H^2 + Q^2L_0^2}\,\tilde{\tau} = -am_0\omega^2\sqrt{H^2 + Q^2L_0^2}.$$

From

$$\phi(L,t) = \phi_0(L_0) + \ell\phi_0'(L_0) + \psi(L_0) + \ldots = 0$$

it follows that

$$\ell = -\psi(L_0)/\phi_0'(L_0).$$

From

$$y(L) = D + a\cos\omega t + \int_0^{L_0+\ell} \sin\phi_0 \,\mathrm{d}s' + \int_0^{L_0} \psi\cos\phi_0 \,\mathrm{d}s' + \ldots = 0$$

it follows that

$$\int_0^{L_0} \tilde{\psi}\cos\phi_0 \,\mathrm{d}s' = -a$$

which is, using (11b), (14) and (12b), equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big[\tilde{\tau}\sin(\phi_0) + H\ \tilde{\psi}\Big] = 0 \qquad \text{at } s = L_0$$

or

$$H\,\tilde{\psi}' + \frac{Q}{H}\,\tilde{\tau} = 0 \qquad \text{at } s = L_0.$$

Summarizing, we have the following boundary conditions:

$$QL_0\,\tilde{\psi} + \frac{H}{\sqrt{H^2 + Q^2L_0^2}}\,\tilde{\tau} = 0 \qquad \text{at } s = 0, \tag{16a}$$

$$QL_0\Big[L_0\,\tilde{\psi}' - \tilde{\psi}\Big] + \frac{H}{\sqrt{H^2 + Q^2L_0^2}}\Big[L_0\,\tilde{\tau}' + \frac{Q^2L_0^2}{H^2 + Q^2L_0^2}\tilde{\tau}\Big] = 0 \qquad \text{at } s = 0, \tag{16b}$$

$$H\sqrt{H^2 + Q^2L_0^2}\,\tilde{\psi}' - QL_0\,\tilde{\tau}' + \frac{H^2Q}{H^2 + Q^2L_0^2}\,\tilde{\tau} + am_0\omega^2\sqrt{H^2 + Q^2L_0^2} = 0 \qquad \text{at } s = 0, \tag{16c}$$

$$H\,\tilde{\psi}' + \frac{Q}{H}\,\tilde{\tau} = 0 \qquad \text{at } s = L_0. \tag{16d}$$

We now nondimensionalize the equation (13), (14) and the systems (11a)-(11b) by scaling

$$s = L_0 \hat{s},\ T_0 = H\hat{T}_0,\ \tilde{\tau} = H\hat{\tau},\ \mu = \frac{QL_0}{H},\ a = L_0\hat{a},\ \Omega = \sqrt{\frac{m_0}{H}}\omega L_0,$$

then the nondimensional form of the static solution along the interval $[0,1]$ becomes

$$\phi_0 = \arctan(\mu(\hat{s}-1)),$$
$$\hat{T}_0 = \sqrt{1+\mu^2(\hat{s}-1)^2},$$

and the dynamic equations are

$$\left[\hat{\tau}\cos\phi_0 - \mu(\hat{s}-1)\tilde{\psi}\right]'' - \Omega^2\tilde{\psi}\sin\phi_0 = 0, \tag{17a}$$
$$\left[\hat{\tau}\sin\phi_0 + \tilde{\psi}\right]'' + \Omega^2\tilde{\psi}\cos\phi_0 = 0, \tag{17b}$$

with boundary conditions

$$\mu\tilde{\psi} + \frac{1}{\sqrt{1+\mu^2}}\hat{\tau} = 0 \qquad \text{at } s = 0, \tag{18a}$$
$$\mu[\tilde{\psi}' - \tilde{\psi}] + \frac{1}{\sqrt{1+\mu^2}}\,\hat{\tau}' + \frac{\mu^2}{(1+\mu^2)^{3/2}}\,\hat{\tau} = 0 \qquad \text{at } s = 0, \tag{18b}$$
$$\sqrt{1+\mu^2}\,\tilde{\psi}' - \mu\hat{\tau}' + \frac{\mu}{1+\mu^2}\,\hat{\tau} + \hat{a}\Omega^2\sqrt{1+\mu^2} = 0 \qquad \text{at } s = 0, \tag{18c}$$
$$\tilde{\psi}' + \mu\hat{\tau} = 0 \qquad \text{at } s = 1. \tag{18d}$$

3 Solution

Since the equations (17a) and (17b) are still complicated, we solved the solution by numerical approximation in MATLAB. We used here central difference discretisation [13]. A MATLAB script file is given in the Appendix.

Furthermore, by assuming μ small, an analytical approximation can be found. Since we may anticipate the fact that τ scales on QL rather than H, it is reasonable, and indeed necessary, to assume $\hat{\tau} = O(\mu)$, so write $\tilde{\tau} = \mu\tau_1$. Since

$$\begin{aligned}
\cos(\phi_0) &= 1 + O(\mu^2), \\
\sin(\phi_0) &= \mu(\hat{s} - 1) + O(\mu^3), \\
\frac{1}{\sqrt{1+\mu^2}} &= 1 + O(\mu^2), \\
\frac{\mu^2}{(1+\mu^2)^{\frac{3}{2}}} &= \mu^2 + O(\mu^4), \\
\sqrt{1+\mu^2} &= 1 + O(\mu^2), \\
\frac{\mu}{1+\mu^2} &= \mu + O(\mu^3)
\end{aligned}$$

then (17a), (17b) can be written as

$$\begin{aligned}
\left[\mu\tau_1 - \mu(\hat{s}-1)\tilde{\psi}\right]'' - \mu(\hat{s}-1)\Omega^2\tilde{\psi} = 0, \\
\left[\mu^2(\hat{s}-1)\tau_1 + \tilde{\psi}\right]'' + \Omega^2\tilde{\psi} = 0
\end{aligned}$$

or

$$\tau_1'' - 2\tilde{\psi}' = 0, \tag{19a}$$
$$\tilde{\psi}'' + \Omega^2\tilde{\psi} = 0. \tag{19b}$$

The boundary conditions become

$$\tau_1 + \tilde{\psi} = 0 \qquad \text{at } s = 0, \tag{20a}$$
$$\tau_1' + \tilde{\psi}' - \tilde{\psi} = 0 \qquad \text{at } s = 0 \tag{20b}$$
$$\tilde{\psi}' + \hat{a}\Omega^2 = 0 \qquad \text{at } s = 0 \tag{20c}$$
$$\tilde{\psi}' = 0 \qquad \text{at } s = 1. \tag{20d}$$

We get the following solution for $\tilde{\psi}$ and $\tilde{\tau}$,

$$\tilde{\psi} = -\frac{\hat{a}\Omega}{\sin\Omega}\cos(\Omega(1-\hat{s})), \tag{21a}$$

$$\tilde{\tau} = \mu H \hat{a}\left[\frac{2}{\sin\Omega}\sin(\Omega(1-\hat{s})) + (\hat{s}+1)\Omega\cot\Omega + \Omega^2\hat{s} - 2\right]. \tag{21b}$$

Note that resonance occurs for values of ω that correspond to $\sin\Omega = 0$, so $\Omega = 0, \pi, 2\pi$, etc.

4 Results

For the examples considered here we used the following typical values: horizontal force H between 1500 kN and 25000 kN, the depth of the water D is 150 m, the weigth of pipe per length Q is 0.978 kN/m, the mass per unit length m_0 is 0.1 kg/m, the frequency ω between 0.2 and 0.46 Hz, and the vertical displacement at the upper end of pipe a is 5 m. From (15) and $\mu = QL_0/H$ we can calculate μ and L_0. For larger values of ω a smaller value of a should be selected to leave the linearisation valid. Of course, the present results would remain the same, apart from a factor, because the equations are linear in a, and all results scale on a.

In Figure (2), (3), (4), and (5), the numerical approximations and the analytical approximation are successively presented by dashed and solid lines. Figure (2) shows the solutions for $\mu = 0.11$. It shows that the analytical approximation is quite close to the numerical solutions. For larger values of μ (see Fig.(4), Fig.(5)), the analytical approximation becomes less accurate, although the agreement is still surprisingly good for relatively large values like μ=0.46 .

5 Acknowledgements

We gratefully acknowledge the enthusiastic help of Bastian van het Hof with the numerical solution in MATLAB.

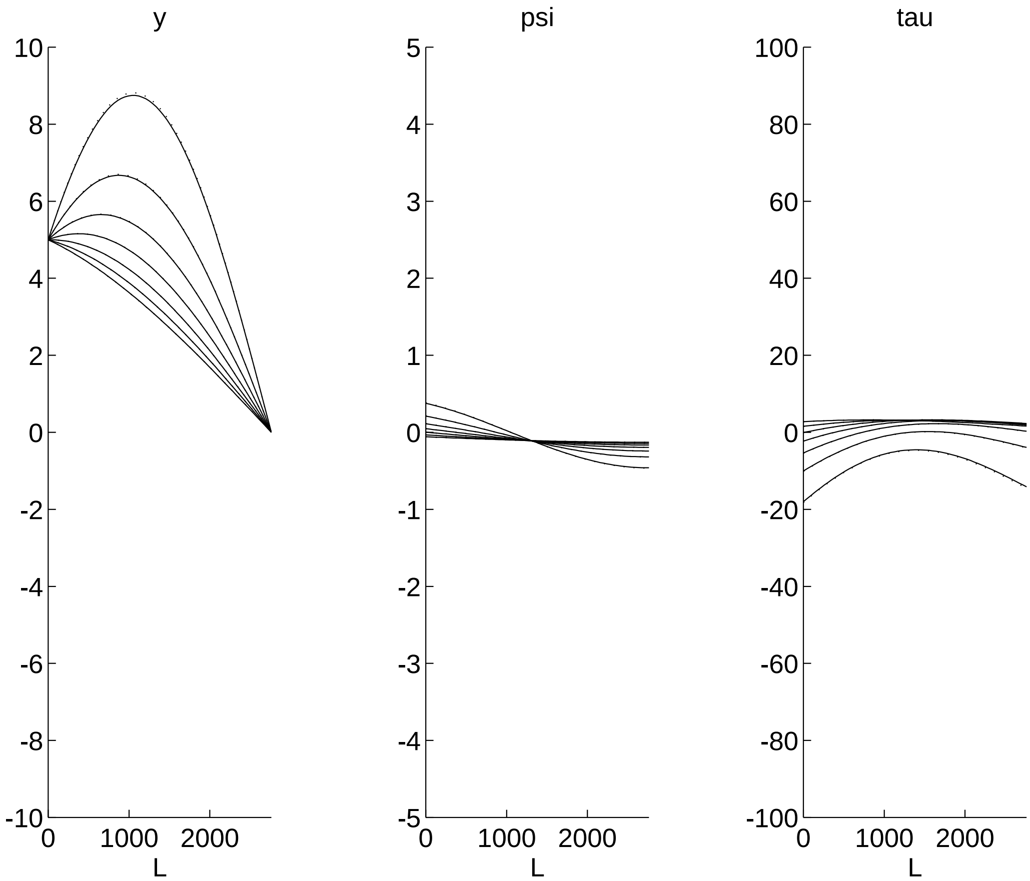

Figure 2: For μ=0.11, with ω=0.20, 0.24, 0.29, 0.33, 0.37, 0.42, 0.46.

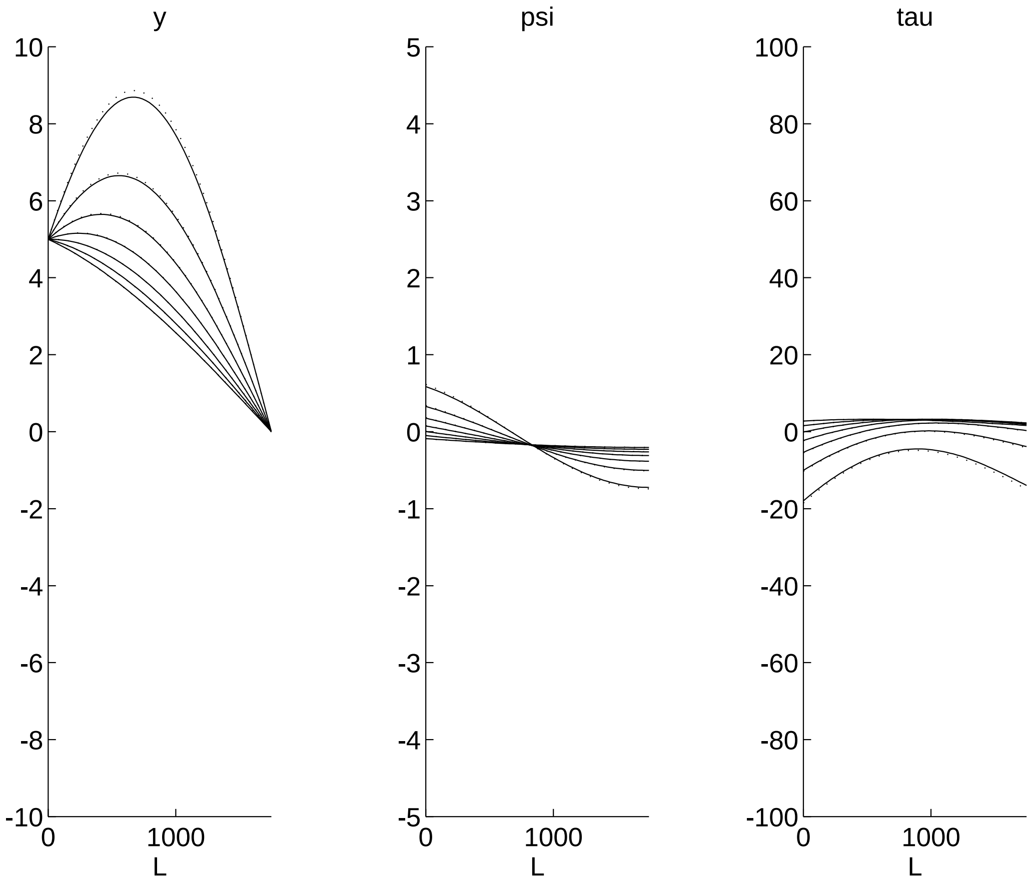

Figure 3: For μ=0.17, with ω=$0.20, 0.24, 0.29, 0.33, 0.37, 0.42, 0.46$.

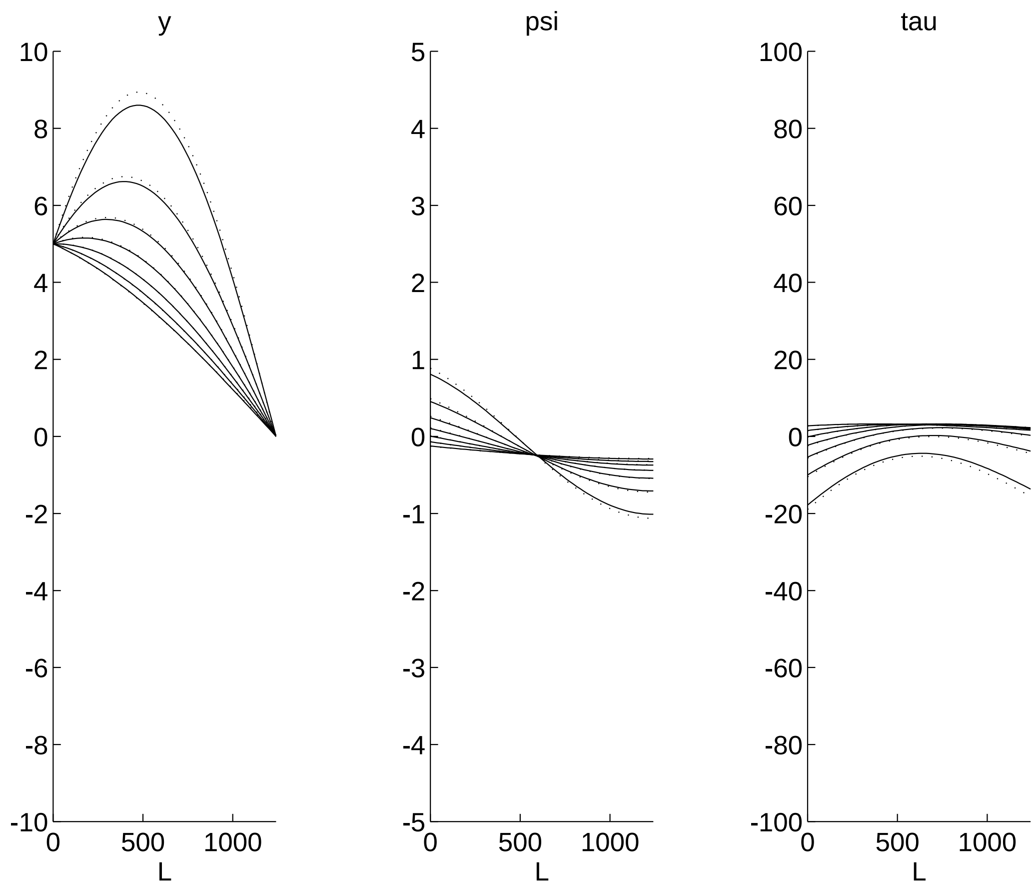

Figure 4: For μ=0.25, with ω=0.20, 0.24, 0.29, 0.33, 0.37, 0.42, 0.46.

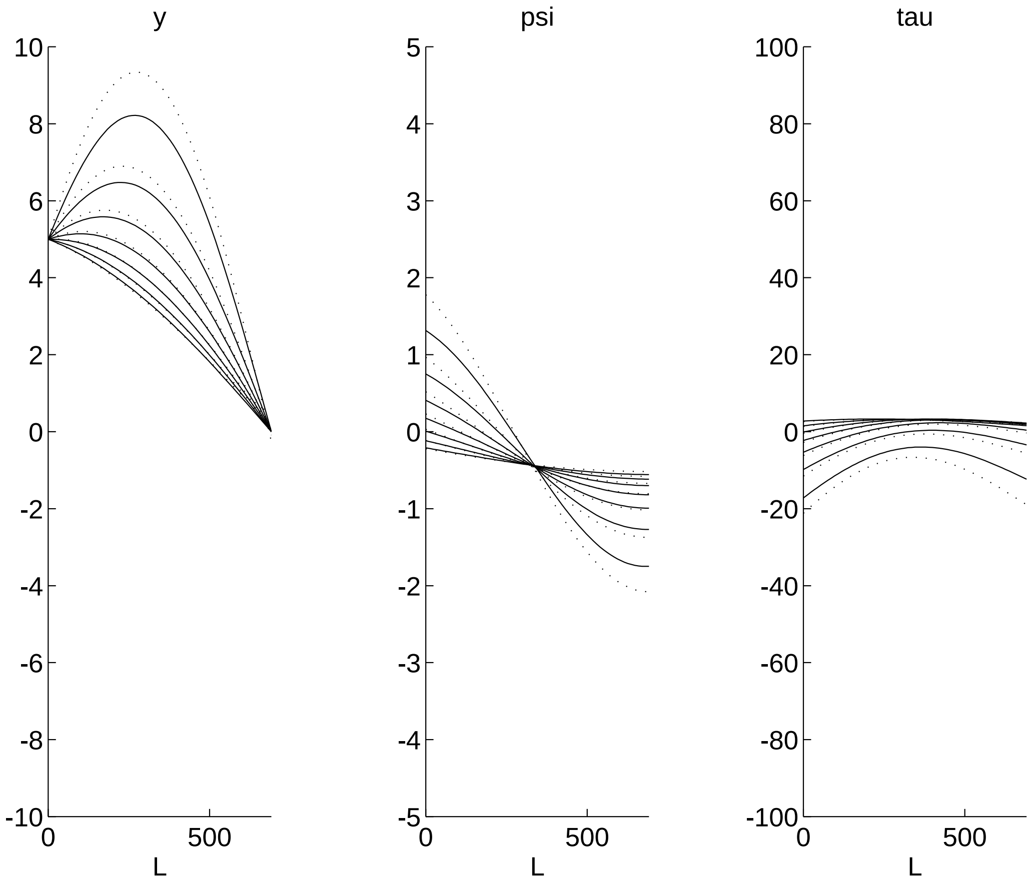

Figure 5: For μ=0.46, with ω=0.20, 0.24, 0.29, 0.33, 0.37, 0.42, 0.46.

Appendix

MATLAB script file of the numerical solution.

```
function cabledyn(om1,om2,nom)
% cable dynamics: motion of suspended flexible cable, on 1 side free
% plots perturbation angle psi and perturbation tension tau
% for nom circular frequencies between om1 and om2

subplot(1,3,1);clg
subplot(1,3,2);clg
subplot(1,3,3);clg
ymax =  10;
pmax =   5;
tmax = 100;

clrtype = str2mat('w-y-c-g-m-r-b-');
clatype = str2mat('w:y:c:g:m:r:b:');

   if nom==1
      omstp = om1;
   else
      omstp = (om2-om1)/(nom-1);
   end

   H  = 5000              ;
   D  = 150               ;
   Q  = 0.987             ;
   m  = 0.1               ;
   a  = 5                 ;
   N  = 2^6               ;
   L  = sqrt(D^2 + 2*D*H/Q);
   mu = Q*L/H             ;
   aa = a/L               ;
   h  = 1/N               ;
   s  = [-h:h:1+h]'       ;
   u  = ones(N+3,1)       ;
   z  = zeros(N+3,1)      ;
```

```
    Phi = atan(mu*(s-1))         ;
    T   = sqrt(1 + mu^2*(s-1).^2);

    % boundary conditions of the type:
    % x1 psi' + x2 psi + x3 tau' + x4 tau = bvj
    v2 = 1+mu^2; v1 = sqrt(v2); v3 = v1*v2;
    a1 = 0 ; a2 =  mu; a3 = 0   ; a4 = 1/v1;     bv1 = 0;
    b1 = mu; b2 = -mu; b3 = 1/v1; b4 = mu^2/v3;  bv2 = 0;
    c1 = v1; c2 =  0 ; c3 =-mu  ; c4 = mu/v2;   %bv3 is omega-dependent
    d1 = 1 ; d2 =  0 ; d3 = 0   ; d4 = mu;       bv4 = 0;

    % and I substitute:
    bv        = zeros(2*N+6,1);
    bv(1)     = bv1;
    bv(N+3)   = bv2;     %bv(N+4) is omega-dependent
    bv(2*N+6) = bv4;

   UV = [spdiags(-mu*(s-1),0,N+3,N+3)  spdiags(cos(Phi),0,N+3,N+3)
         spdiags(  u      ,0,N+3,N+3)  spdiags(sin(Phi),0,N+3,N+3)];

   % so I have : u = - mu*(s-1).*psi + cos(Phi).*tau;
   %              v =          1.*psi + sin(Phi).*tau;

   Dxx = spdiags( [u;u]*[1,-2,1]/h^2 , [-1,0,1] , 2*N+6 , 2*N+6 );

   A1  = Dxx*UV;

   B = [ spdiags( sin(Phi),0,N+3,N+3) spdiags(z,0,N+3,N+3)
         spdiags(-cos(Phi),0,N+3,N+3) spdiags(z,0,N+3,N+3) ];

 for j = [1:nom]    % for a number of omega's

    if nom==1
      omega = om1;
    else
      omega = om1+(j-1)*(om2-om1)/(nom-1);
    end
```

```
 Omega = sqrt(m/H)*omega*L;
 bv3   = -aa*Omega^2*v1;

 A     = A1 - Omega^2*B;

%   [- mu*(s-1).*psi + cos(Phi).*tau]'' =  Omega^2*sin(Phi).*psi
%   [         1.*psi + sin(Phi).*tau]'' = -Omega^2*cos(Phi).*psi

% make some room for boundary conditions (A is sparse!):
A([1,N+3,N+4,2*N+6],:) = zeros(4,2*(N+3));
A(    1,[    1,    3]) = a1*[-1,1]/(2*h) ; %psi'(0)
A(    1,[  N+4,  N+6]) = a3*[-1,1]/(2*h) ; %tau'(0)
A(    1,[    2,  N+5]) = [a2,a4]         ; %psi(0),tau(0)
A(  N+3,[    1,    3]) = b1*[-1,1]/(2*h) ; %psi'(0)
A(  N+3,[  N+4,  N+6]) = b3*[-1,1]/(2*h) ; %tau'(0)
A(  N+3,[    2,  N+5]) = [b2,b4]         ; %psi(0),tau(0)
A(  N+4,[    1,    3]) = c1*[-1,1]/(2*h) ; %psi'(0)
A(  N+4,[  N+4,  N+6]) = c3*[-1,1]/(2*h) ; %tau'(0)
A(  N+4,[    2,  N+5]) = [c2,c4]         ; %psi(0),tau(0)
A(2*N+6,[  N+1,  N+3]) = d1*[-1,1]/(2*h) ; %psi'(1)
A(2*N+6,[2*N+4,2*N+6]) = d3*[-1,1]/(2*h) ; %tau'(1)
A(2*N+6,[  N+2,2*N+5]) = [d2,d4]         ; %psi(1),tau(1)

bv(N+4)     = bv3;

% now solve the system:
y   = A\bv;                    % gives the same answer as inv(A)*bv;
psi =   y(  2:  N+2);
tau = H*y(N+5:2*N+5);

% the analytical approximation
sa  =  s(2:N+2);
psa = -aa*Omega/sin(Omega)*cos(Omega*(1-sa));
taa =  H*mu*aa*(2/sin(Omega)*sin(Omega*(1-sa))+...
                Omega*cot(Omega)*(sa+1)+Omega^2*sa-2);

% the y-coordinate:
yy =  zeros(N+1,1);
ya =  zeros(N+1,1);
```

```
yy(1) = aa;
ya(1) = aa;
for i = [2:N+1]
    yy(i) = yy(i-1) + (psi(i-1)*cos(Phi(i))+psi(i)*cos(Phi(i+1)))*h/2;
    ya(i) = ya(i-1) + (psa(i-1)*cos(Phi(i))+psa(i)*cos(Phi(i+1)))*h/2;
end

% and plot the results:
jj = 1+rem(j-1,7);
subplot(1,3,1); hold on;
                plot(sa*L,yy*L,clrtype(2*jj-1:2*jj));
                plot(sa*L,ya*L,clatype(2*jj-1:2*jj));
                title('y');
                axis([0,L,-ymax,ymax])
                hold off;
subplot(1,3,2); hold on;
                plot(sa*L,psi*180/pi,clrtype(2*jj-1:2*jj));
                plot(sa*L,psa*180/pi,clatype(2*jj-1:2*jj));
                title('psi');
                axis([0,L,-pmax,pmax])
                hold off;
subplot(1,3,3); hold on;
                plot(sa*L,tau,clrtype(2*jj-1:2*jj));
                plot(sa*L,taa,clatype(2*jj-1:2*jj));
                title('tau');
                axis([0,L,-tmax,tmax])
                hold off;
fprintf(1,...
'%2i:  Om=%6.2f,  mu=%6.2f,  max psi=%7.2f,  max y=%7.2f, max t=%9.2f\n',...
 j, Omega,mu,max(abs(psi)*180/pi),max(abs(yy*L)),max(abs(tau)));

end
```

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