Another form of I is given by

$$I=4\pi\int_0^\infty J_0(r\sigma)\frac{\sin(t\sigma\sqrt{\sigma^2+a^2})}{\sqrt{\sigma^2+a^2}}\,d\sigma.$$

An Inverse Potential Problem

Problem 88-15, *by* M. S. KLAMKIN (University of Alberta).

The following is a known result and is given as a problem in [1]:

"Current enters an infinite plane conducting sheet at some point P and leaves at infinity. A circular hole, which does not include P, is cut anywhere in the sheet. Show that the potential difference between any two points on the edge of the hole is twice what it was between the same two points before the hole was cut."

(i) Show that the above result can be extended to a general two-dimensional irrotational flow of an inviscid incompressible fluid.

(ii)* Prove or disprove that only a circular hole has the above property.

(iii) Show that in three dimensions, the analogous result for a sphere does not hold if the undisturbed flow is due to a source, but it does hold if the potential of the undisturbed flow is a spherical harmonic of degree n.

REFERENCE

[1] W. R. SMYTHE, *Static and Dynamic Electricity*, McGraw–Hill, New York, 1939, p. 253, #51.

SOLUTIONS

Monotonicity of Bessel Functions

Problem 87-11*, *by* S. W. RIENSTRA (Katholieke Universiteit, Nijmegen, the Netherlands).

The eigenvalue equation related to a problem of sound propagation in hard-walled annular ducts is given by

$$x^2\{J_n'(x)Y_n'(xh)-Y_n'(x)J_n'(xh)\}=0$$

with $n=0, 1, 2, \cdots, 0<h<1$, where J_n' and Y_n' denote derivatives of Bessel functions, while h is the ratio between the inner and outer duct radii. We are interested in the solutions $x=\alpha(h)$ as a function of h. In particular, we want to show that

$$\frac{d\alpha}{dh}=\frac{\alpha f_n(\alpha)}{h\{f_n(\alpha h)-f_n(\alpha)\}}\quad\text{where}\quad f_n(x)=\frac{J_n'(x)^2+Y_n'(x)^2}{1-n^2/x^2},$$

is always finite or $f_n(\alpha h)-f_n(\alpha)\neq 0$.

If $n=0$, the latter is true since $f_0(x)=J_1(x)^2+Y_1(x)^2$ is a decreasing function [1]. It is also true for $n\geqq 1$ if $\alpha h<n$ and $\alpha>n$ ($\alpha\leqq n$ does not occur). If $\alpha h=n$, $d\alpha/dh=0$. Finally, it will also be true for the case $\alpha h>n$ if $f_n(x)$ is decreasing for $x>n$.

In view of numerical evidence that this is so for $n=0, 1, \cdots, 100$, it is conjectured to be true. Prove or disprove.

REFERENCE

[1] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, Cambridge Univ. Press, New York, 1948, p. 446.

Solution by W. B. JORDAN (Scotia, New York).

We write y for hx. The given eigenvalue equation is

(1) $$J_n'(x)Y_n'(y)-J_n'(y)Y_n'(x)=0.$$

Its derivative with respect to x is

$$J_n''(x)Y_n'(y)+y'J_n'(x)Y_n''(y)-y'J_n''(y)Y_n'(x)-J_n'(y)Y_n''(x)=0.$$

We simplify this expression by using (1), the Wronskian

$$J_{n+1}(t)Y_n(t)-J_n(t)Y_{n+1}(t)=2/\pi t$$

and the recurrence formulas

$$C_n'(t)=C_{n-1}(t)-\frac{n}{t}C_n(t)=-C_{n+1}(t)+\frac{n}{t}C_n(t)$$

in which C may be either J or Y. We get

$$(a)+\frac{n}{x^2}(b)+y'\left((c)+\frac{n}{y^2}(d)\right)=0$$

where

$$\begin{aligned}(a)&=J_n'(y)Y_{n+1}'(x)-J_{n+1}'(x)Y_n'(y)\\&=\frac{2}{\pi x}\left[\frac{n(n+1)}{x^2}-1\right]\frac{J_n'(y)}{J_n'(x)},\\(b)&=J_n'(y)Y_n(x)-J_n(x)Y_n'(y)=-\frac{2}{\pi x}\frac{J_n'(y)}{J_n'(x)},\\(c)&=J_{n+1}'(y)Y_n'(x)-J_n'(x)Y_{n+1}'(y)\\&=-\frac{2}{\pi y}\left[\frac{n(n+1)}{y^2}-1\right]\frac{J_n'(x)}{J_n'(y)},\\(d)&=J_n(y)Y_n'(x)-J_n'(x)Y_n(y)=\frac{2}{\pi y}\frac{J_n'(x)}{J_n'(y)}\end{aligned}$$

so

$$\frac{1}{x}\left(\frac{n^2}{x^2}-1\right)J_n'^2(y)-\frac{y'}{y}\left(\frac{n^2}{y^2}-1\right)J_n'^2(x)=0.$$

Now $y'=h+xh'$, so for dx/dh to be ∞ it is necessary that $h'=0$, which can occur only if

(2) $$\frac{x^2}{x^2-n^2}J_n'^2(x)=\frac{y^2}{y^2-n^2}J_n'^2(y).$$

We are to show that (1) and (2) cannot both be true. In the notation of [1, §9.2], namely,

$$J'_n(t)=N_n(t)\cos\phi_n(t), \qquad Y'_n(t)=N_n(t)\sin\phi_n(t),$$

(1) gives

$$\sin[\phi_n(y)-\phi_n(x)]=0, \qquad \phi_n(x)=\phi_n(y)+k\pi,$$

k being some integer; and (2) becomes

$$f_n(x)=f_n(y)$$

where

$$f_n(t)=t^2N_n^2(t)/(t^2-n^2)=2/\pi t\phi'_n(t)$$

on using (9.2.21). For t large, $N_n^2(t)=2/\pi t$, so $\phi'_n(t)=1-n^2/t^2$; thus $\phi'_n(n)=0$ and $\phi'_n(\infty)=1$. For $n \geqq 2$ and $t \geqq n$ the formula $1-n^2/t^2$ is a decent first approximation to $\phi'_n(t)$, so ϕ'_n is monotone increasing and f_n is monotone decreasing. It follows that $f_n(x)$ cannot equal $f_n(y)$ if $x \neq y$.

REFERENCE

[1] M. ABRAMOWITZ AND I. STEGUN, *Handbook of Mathematical Functions*, National Bureau of Standards, Washington, D.C., 1965.

Also solved by J. A. COCHRAN (Washington State University) who refers to results in his paper "The analyticity of cross-product Bessel function zeros," *Proc. Cambridge Philos. Soc.*, 62 (1966), pp. 215–256.

A Nonnegative Integral

Problem 87-13, *by* J. GEVIRTZ *and* O. G. RUEHR (Michigan Technological University), Houghton, Michigan).

For $t>1$, show that $F(t)\geqq 0$, where

$$F(t)=\int_0^1\left\{z\ln\left(\frac{1+z}{1-z}\right)-\frac{t(z^2+1)}{1+t^2}\ln\left(\frac{2}{1-z^2}\right)\right\}\frac{dz}{t^2-z^2}.$$

The problem arose in connection with the development of sharp mean value theorems in the unit disk.

Solution by M. L. GLASSER (Clarkson University, Potsdam, New York).

To simplify matters we map the region $t>1$ onto the (open) unit interval by the substitution $u=(t-1)/(t+1)$. Then straightforward integration [1] gives

$$F(t)=(2-\ln 2)(u^2+1)^{-1}-\tfrac{1}{2}\ln 2\ln u+\ln u\ln(1-u)+\mathrm{Li}_2(u)$$
$$-1+\frac{\pi^2}{6}-\frac{1}{2}\ln 2\equiv\phi(u)$$

where Li_2 denotes the dilogarithm. We have

$$\phi(u)\cong\tfrac{1}{2}\ln 2\ln u \quad \text{as } u\to 0^+ \quad (t\to 1^+),$$

$$\phi(1^-)=0 \quad (t\to\infty)$$