Modelling and Perturbation Methods

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Chapter I

The art of modelling

The mathematical solution of a “real world” problem starts with the modelling phase, where the problem is described in a mathematical representation of its primitive elements and their relations. As the solution is not served by unnecessary complexity, we are interested in an adequate mathematical description with the lowest number of essentially independent parameters and variables.

A very important aspect in the modelling is therefore the identification of a hierarchy of importance, to distinguish between the important, the less important, and the unimportant effects. From this hierarchy we may decide which aspects can be included and which can be neglected in the model.

1 Introduction

Modelling in the context of applied mathematics is something like

*Describing a real-world problem in a mathematical way by what is called a MODEL, such that it becomes possible to deploy mathematical tools for its solution. The accuracy of the description should be limited, in order to make the model not unnecessarily complex. The model should be based on first principles and elementary relations, such that it has reasonable claims to predict both quantitative and qualitative aspects of the solution.*

This is evidently a very loose definition. Apart from the question what is meant with: a problem being described in a mathematical way, there is the confusing paradox that we only know the precision of our model, if we can compare it with a better model, but this better model is exactly what we try to avoid as it is usually unnecessarily complex! In general we don’t know the problem and quality of its accompanying model well enough to be absolutely sure that the sought description is both consistent, complete and sufficiently accurate for the purpose, and not too formidable for any treatment. A model is, therefore, to a certain extent a vague concept.

On the other hand, however, modelling plays a key rôle in applied mathematics, since mathematics cannot be applied to any real world problem without the intermediate steps of modelling. So modelling is arguably the main or nearly the main *raison d’être* of applied mathematics. Therefore, a more structured approach is necessary, which is the aim of the present chapter.

Some people define *modelling* as the process of translating a real-world problem into mathematical terms. We will not do so, as this definition is too wide to include the subtle aspects of “limited precision”.
2. MODELS

A model may be too crude, but also it may be too refined. It is too crude if it just doesn’t describe the problem considered, or if the numbers it produces are not accurate enough to be acceptable. It is too refined if it includes irrelevant effects that make the problem untreatable, or make the model so complicated that important relations or trends remain hidden.

Therefore we will introduce the word “mathematizing”, defined as the process of translating a real-world problem into mathematical terms. It is a translation in the sense that we translate from the inaccurate, verbose “everyday” language to the language of mathematics. For example, the geometrical presence of objects in space may be described parametrically in a suitable coordinate system. In a similar way, any properties that are known and expected to play a rôle may be formulated, in some suitable accuracy, by a field or function in time and/or space, explicitly or implicitly, for example as a differential equation.

A very important point to note is the fact that such a formulation is always at some level simplified. The earth can be a point or a sphere in astronomical applications, or an infinite half-space or just absent in problems of human scale. Based on the level of simplification, or, put in another way, of sophistication or accuracy, we can associate an inherent hierarchy to the set of possible descriptions.

Mathematizing is an elementary but not trivial step. In fact, it forms the single most important step in the progress of science. It requires the distinction, naming, and exact specification of the essential relevant elementary objects and their interrelations, where mathematics acts as a language in which the problem is described. If theory is available for the mathematical problem obtained this way, the problem considered may be subjected to the strict logic of mathematics, and reasoning in this language will transcend over the limited and inaccurate ordinary language.

Mathematizing is therefore not only important scientifically in general, it is also important from a more restricted mathematical point of view, because it provides the link between the mathematical world and the real world. This is important for mathematics in general because the most important, most challenging, and therefore probably most interesting mathematical problems are those originating from a real-world problem.

The ultimate goal for mathematizing a problem is a deeper understanding and a more profound analysis and solution of the problem. Now a refined translation is usually more accurate but also more complicated and more difficult – if not impossible! – to analyse and solve than a simpler one. Therefore, not every mathematical translation is a good one.

We will call a good mathematical translation a model or mathematical model if it is lean in the sense, that it describes our problem quantitatively or qualitatively in a suitable or required accuracy with a minimal number of essentially different parameters and variables. (We say “essentially different”, in view of a reduction that is always possible by writing the problem in dimensionless form. See Buckingham’s theorem I.7 below.)

Again, this definition is rather subjective, as it greatly depends on the context of the problem considered and our knowledge and resources. So there will rarely be one “best” model. At the same time, it shows that modelling, even if relying significantly on intuition, is part of the mathematical analysis.

2 Models

We will distinguish the following three classes of models.
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• Systematic models.
Other possible names are reducing models or asymptotic models. Here one starts to use available complete models, which are adequate, but so over-complete that effects are included which are irrelevant, uninteresting, or negligibly small, and thereby making the mathematical problem unnecessarily complex. By using available additional information (order of magnitude of the parameters) assumptions can be made which minimize in a systematic way the over-complete model into a good model: small parameters are taken zero, large parameters become infinite, an almost symmetry becomes a full symmetry.

Example I.1 Examples of asymptotic models are numerous. Any ordinary flow is (usually) described by a model which is reduced from the full (i.e. compressible, viscous) Navier-Stokes equations. For the sake of argument we take a simple convection-diffusion problem. For a temperature field $T$ and given velocity field $v$ in space $x$ and time $t$ and diffusion coefficient $\alpha$ we have the assumed “over-complete” model

$$\frac{\partial T}{\partial t} + v \cdot \nabla T = \alpha \nabla^2 T,$$

(2.1)

which is, however, difficult to solve. If we have reasons to assume that the diffusion term may be ignored, we obtain the reduced problem

$$\frac{\partial T}{\partial t} + v \cdot \nabla T = 0,$$

(2.2)

which is far more attractive than the full problem, as it may be solved exactly along the streamlines as

$$\frac{d}{dt} T(\xi(t), t) = 0 \rightarrow T(\xi(t), t) = \text{constant}, \quad \text{where } \frac{d\xi}{dt} = v. \quad (2.3)$$

• Constructing models
Other possible names are building block models or lumped parameter models. Here we build our problem description step by step from low to high, from simple to more complex, by adding effects and elements like building blocks, until the required accuracy or adequacy is obtained.

Example I.2 The following problem of air release by a simple air pump may be an example of a building block model. Consider a pump of cross section $S$ and length $a(t)$, which depends on the piston position. Under pressure, the enclosed volume of air $Sa(t)$ leaves the pump through a small hole, forming a jet of cross section $S_j$ and (mean) velocity $v_j$. From time $t = 0$, a spring pushes against the piston with a force $F = \lambda a(t)$. Assuming any inertia effects of the piston to be much smaller than the inertia of the flow, the piston force is balanced by a pressure increase from $p_\infty$ to $p_0$ inside the pump. So $F = S(p_0 - p_\infty)$. Assuming incompressible air of density $\rho_0$, conservation of mass tells us that $\{\text{the variation inside}\} = \{\text{what goes out}\}$. In formula this is

$$\frac{d}{dt} \left( \rho_0 a(t) \right) = -\rho_0 v_j S_j \quad \rightarrow \quad v_j = \frac{S}{S_j} \frac{da}{dt}. \quad (2.4)$$

From Bernoulli’s law, relating pressure $p$ and velocity $v$ by $p + \frac{1}{2} \rho_0 v^2 = \text{constant}$, and noting that the pressure inside the jet is equal to the atmospheric pressure $p_\infty$, we can deduce that

$$p_0 = p_j + \frac{1}{2} \rho_0 v_j^2 = p_\infty + \frac{1}{2} \rho_0 v_j^2 \quad \rightarrow \quad \frac{1}{2} \rho_0 v_j^2 = \frac{\lambda a}{S}. \quad (2.5)$$
Together we have the model
\[ \frac{da}{dt} = -K \sqrt{a}, \quad \text{with} \quad K = \frac{S_j}{S} \sqrt{\frac{2\lambda}{\rho_0 S}} \] (2.6)

which is easily solved by
\[ a(t) = L \left( 1 - \frac{1}{2} \frac{K t}{\sqrt{L}} \right)^2 \quad \text{along} \quad 0 \leq t \leq \frac{2\sqrt{L}}{K}. \] (2.7)

\section*{Canonical models}

Other possible names are characteristic models or essential models. Here an existing model is further reduced to describe only the essence of a certain aspect of the problem considered. These models are particularly important if the mathematical analysis of a model from one of the other categories is lacking available theory. The development of such theory is usually hindered by too much irrelevant details. These models are useful for the understanding, but usually far away from the original full problem setting and therefore not suitable for direct industrial application.

\textbf{Example I.3} The Navier Stokes equation describing the behaviour of incompressible viscous flow in terms of a velocity and pressure field is in general very complex.
\[ \frac{\partial}{\partial t} v + v \cdot \nabla v = -\frac{1}{\rho} \nabla p + \nu \nabla^2 v \] (2.8)

Especially the coupling between the non-linear and viscous terms yielding instabilities and turbulence is complicated and difficult to analyse. Therefore, Burgers proposed to consider the following very simplified version of it, where the pressure gradient has been neglected, and only behaviour in one dimension is taken into account. This equation
\[ \frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u = \nu \frac{\partial^2}{\partial x^2} u \] (2.9)
is called Burgers’ equation [36]. A great deal of insight is obtained by the remarkable transformation
\[ u = -2v\phi^{-1} \frac{\partial}{\partial x} \phi \] (2.10)
found independently by Cole (1951) and Hopf (1950), by which the nonlinear equation is reduced to a linear equation, related to the heat equation
\[ \frac{\partial}{\partial t} \phi - \nu \frac{\partial^2}{\partial x^2} \phi = C(t)\phi \] (2.11)
where \( C(t) \) is an arbitrary function of \( t \). This equation is well understood and allows many exact solutions.

Note that an asymptotic model may start as a building-block model, which is only found at a later stage to be too comprehensive. Similarly, a canonical model may reduce from an asymptotic model if the latter appears to contain a particular, not yet understood effect, which should be investigated in isolation before any progress with the original model can be made.
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3 Non-dimensionalization and scaling

Modelling implies to decide which effects are relevant and should be included, and which are irrelevant and can be ignored. More in general, we may expect a hierarchy in relevance, from most dominant, via less relevant and locally irrelevant to absolutely unimportant effects or contributions.

Relevant and irrelevant are rather vague qualifications. To make this operational we will relate them to small and large terms in our mathematical description (equations, etc.).

Small and large have no absolute meaning, as long as we have not defined our “measuring stick”. To illustrate this we may imagine the following science fiction scenery. Suppose we are lost in outer space, with all planets, stars, and galaxies so far away that they are only seen as sizeless spots on our retina. Then, a rock drifts slowly into our field of vision. As long as we are not close enough for a stereoscopic view with both our eyes, we are not able to compare its size or distance with anything we know. There is no way to estimate if it is big and far away, or small and nearby. Only the rock itself is our scale of reference.

A similar experience is found when we look into a microscope of unknown amplification. An object, visible but not recognizable, may be as big as an amoeba, or as small as a virus or a molecule. Re-interpreting the famous saying of Protagoras, *Man is the measure of all things*, nothing we observe is small or large, fast or slow, in any absolute sense. It is only by comparison that these qualifications have a meaning.

The next question is: what do we use for comparing. We can use an absolute or universal measuring stick, like a meter or a kilogram, if we want to archive the observations in order to be able to reproduce them exactly again. However, we use a natural scale, like typical sizes in the problem itself, if we want to classify the type of phenomena.

Inherent scaling

When we model, we need to understand the problem in advance to a certain degree, such that we are able to formulate the relevant physical laws and relations. Therefore, in modelling the natural scaling is the appropriate one to use. We introduce for all our dependent and independent variables typical values, taken from the problem in question. For example, a length \( L \) for the independent spatial variable \( x \), and a velocity \( V \) for the dependent variable \( v \), and thus an intrinsic time \( L/V \) for time coordinate \( t \).

Dimensionless numbers

When more than one problem parameters in the same units is available, for example a length \( L \) and width \( D \), or a time \( L/V \) and a frequency \( \omega \), it is inevitable that if one is selected for the scaling, the combination with the other gives us a new parameter, like \( D/L \) or \( \omega L/V \), which is now independent of the units (meters, seconds) and is therefore called a dimensionless parameter. Incidentally, this meaning of the word “dimension” has nothing to do with the mathematical meaning of the number of independent basis vectors in a vector space. Dimensionless parameters are very important for a systematic classification of types of problems. They measure the relative importance of certain effects in an absolute way.
3. **NON-DIMENSIONALIZATION AND SCALING**

**Principle of Dimensional Homogeneity**

If the model is a proper one, reflecting the intrinsic relations between the variables, it should not depend on the arbitrary use of meters or inches, etc. Terms which are added or subtracted should have the same units. The equations should be dimensionally homogeneous.

**Example I.4** Consider the following model of a quantity \( x \) satisfying the equation

\[
ax^2 + bx + c = 0. \tag{3.1}
\]

Assume that \( x \) denotes a length, with units in meters, denoted by \([x] = \text{m}\), and \( c \) is a velocity with units in meters per second, or \([c] = \text{m/s}\). If the equation is dimensionally homogeneous with \([ax^2] = [bx] = [c]\), the units of the other parameters \( a \) and \( b \) cannot be else but \([a] = [c]/[x^2] = 1/\text{ms}\) and \([b] = [c]/[x] = 1/\text{s}\). Therefore, we can scale time and length on several combinations as follows

\[
x = \frac{c}{b} X, \quad a = \frac{b^2}{c} \quad \rightarrow \quad \alpha X^2 + X + 1 = 0, \tag{3.2a}
\]

or (if \( ac > 0 \))

\[
x = \sqrt{\left(\frac{c}{a}\right)} X, \quad b = \sqrt{(ac)} \beta \quad \rightarrow \quad X^2 + \beta X + 1 = 0, \tag{3.2b}
\]

or

\[
x = \frac{b}{a} X, \quad c = \frac{b^2}{a} \gamma \quad \rightarrow \quad X^2 + X + \gamma = 0. \tag{3.2c}
\]

The constants \( \alpha, \beta, \gamma \) are dimensionless constants, parametrizing the respective reduced problem. It should be noted that any of these scalings are equivalent (no information is lost), but they are not equally useful. The preferred reduction is the one in which \( x \) is scaled on a value typically occurring in the situation considered, and \( X \) is henceforth of order unity. So a careful inspection of the range of numerical values of \( x \) and the parameters \( a, b, c \) is essential. Only then the dimensionless parameter, finally left, can tell us more about the behaviour of \( X \).

**Example I.5** Consider an object of typical size \( L \) has initially a temperature distribution \( T(x, 0) = T_0(x) \). The temperature \( T \) satisfies a heat diffusion equation with diffusion constant \( \alpha \). The edges of the object are kept at a constant temperature \( T_{\text{edge}} = 0 \). We scale \( x \) on \( L \), the only length scale in the problem. As the problem is linear, it is not really necessary to scale \( T \), but we could use the mean, or maximum value of \( T_0 \). There is no explicit time scale in the problem, for example from an external source. The only parameter with the dimension of time is \( L^2/\alpha \). If we assume a balance between decay and diffusion, the typical decay time (the half-life, say) is just of the order of \( L^2/\alpha \), which is therefore the natural time to scale on.

\[
\frac{\partial T}{\partial t} = \alpha \nabla^2 T, \\
T_{\text{edge}} = 0, \quad T(x, 0) = T_0(x) \\
t = \frac{t_0}{L^2} \\
x = \xi L \}
\]

\[
\frac{1}{t_0} \frac{\partial T}{\partial \tau} = \frac{\alpha}{L^2} \nabla^2 \xi T
\]

\[
\square
\]
**Example I.6** A piece of metal of size $L$ is heated by applying an electric field with potential $\phi$ and typical voltage $V$. This heat source is given $|\nabla \phi|^2$ (see for an extensive description example I.12). From a balance of the storage, dissipation and source terms, it follows that the generated heat is dissipated through the metal with a typical decay time of $O(\rho c L^2/k)$, and the final temperature of the stationary state is typically $O(\sigma V^2/k)$.

\[
\begin{align*}
\frac{\rho c}{\partial t} \frac{\partial T}{\partial t} &= k \nabla^2 T + \sigma |\nabla \phi|^2, & T(x, y, 0) = 0 \\
T &= T_0 \theta \\
t &= t_0 \tau \\
x &= L \xi \\
\phi &= V \psi
\end{align*}
\]

\[
L
\]

\[L / \text{A}^3\]

3.1 Dimensional Analysis

In any description of reality, the variables & parameters have physical dimensions and are therefore dimensionally related. For example, any relation or physical law is only universally valid if the expression or expressions are dimensionally homogeneous, and independent of the physical dimensions used. This fact alone implies that the number of essentially independent groups of variables & parameters is less than the number of variables & parameters.

**Buckingham’s $\Pi$-theorem**

Suppose we have a physical variable $q_0$, described by $n$ variables & parameters: $q_1, q_2, \ldots, q_n$, in $r$ independent physical dimensions (kg, m, sec, A, K, ...): $d_1, d_2, \ldots, d_r$. $[q]$ denotes the dimension, or unit of scale, of $q$. It is not even necessary that there is an explicit model, only some relation like

\[
q_0 = f(q_1, q_2, \ldots, q_n).
\]

In general, $f$ is any combination of algebraical and transcendental expressions. Any transcendental functions like sin and exp are dimensionless, so the argument should be dimensionless too. Hence, we can assume that $f$ can be written as

\[
q_0 = q_1^{\gamma_1} q_2^{\gamma_2} \ldots q_n^{\gamma_n} \Phi(R_1, \ldots, R_m)
\]

where $\Phi$ depends on $m$ dimensionless groups of $q_1, \ldots, q_n$ of the form $R_j = q_1^{a_1} q_2^{a_2} \ldots q_n^{a_n}$.

As the number ($m$) of essentially different dimensionless groups tells us something about the complexity of the model involved (and for a good model this shouldn’t be too high; see the above sections I.1,2), we are interested to determine $m$, i.e. to determine how many dimensionless groups are at most possible.

From the Principle of dimensional homogeneity each group has the same dimension, so

\[
[R_j] = [q_1^{a_1} q_2^{a_2} \ldots q_n^{a_n}] = \cdots = 1.
\]
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From the same Principle the dimension \([q_j]\) of each \(q_j\) may be written as

\[
[q_j] = d_1^{\mu_{1j}} d_2^{\mu_{2j}} \ldots d_r^{\mu_{rj}} = \prod_{i=1}^{r} d_i^{\mu_{ij}}
\]  

(3.6)

So we have

\[
\left[ \prod_{j=1}^{n} q_j^{\alpha_j} \right] = \prod_{j=1}^{n} \prod_{i=1}^{r} d_i^{\mu_{ij} \alpha_j} = \prod_{i=1}^{r} d_i^{\sum_{j=1}^{n} \mu_{ij} \alpha_j} = 1
\]  

(3.7)

In other words, the products \(\prod_{j=1}^{n} q_j^{\alpha_j}\) are the possible dimensionless groups that describe our problem. The number of possible groups is the number of (non-trivial) solutions of

\[
\sum_{j=1}^{n} \mu_{ij} \alpha_j = 0 \quad \text{for } i = 1, 2, \ldots, r \text{ and any } \alpha_j,
\]  

(3.8)

or in matrix notation

\[
\begin{pmatrix}
\mu_{11} & \mu_{12} & \ldots & \mu_{1n} \\
\mu_{21} & \ddots & & \\
\vdots & & \ddots & \\
\mu_{r1} & & \ldots & \mu_{rn}
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_n
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
\]  

(3.9)

Hence, we have an \(n - r\) dimensional (non-trivial) solution space, as the number of homogeneous solutions is equal to the dimension \(n\) of the solution space minus the rank of the matrix \(k\).

In the same way we find for \(q_0\)

\[
[q_0] = d_1^{\mu_{10}} d_2^{\mu_{20}} \ldots d_r^{\mu_{r0}} = \prod_{i=1}^{r} d_i^{\mu_{i0}}
\]  

(3.10)

that the possible \(\gamma_1, \ldots, \gamma_n\) satisfy

\[
\begin{pmatrix}
\mu_{11} & \mu_{12} & \ldots & \mu_{1n} \\
\mu_{21} & \ddots & & \\
\vdots & & \ddots & \\
\mu_{r1} & & \ldots & \mu_{rn}
\end{pmatrix}
\begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\vdots \\
\gamma_n
\end{pmatrix}
= 
\begin{pmatrix}
\mu_{10} \\
\mu_{20} \\
\vdots \\
\mu_{n0}
\end{pmatrix}
\]  

(3.11)

and thus have in general \(n - r + 1\) solutions: one solution of the inhomogeneous problem and \(n - r\) solutions of the homogeneous problem.

This yields Buckingham’s \(\Pi\)-theorem (\(\Pi\) stands for the products).

**Theorem I.7 (Buckingham’s \(\Pi\)-theorem).** If a physical quantity is described by \(n\) variables & parameters in \(r\) dimensions, the number of essentially different problem parameters is at most \(n - r\). The number of possible dimensionless groups, including the primary quantity, is at most \(n - r + 1\). Dimensionless groups consisting of a combination of only problem parameters are called **dimensionless numbers**. Groups consisting of a combination of parameters and several variables are called **similarity variables**.
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Example I.8 Consider the drag \( D \) - the reaction force due to the surrounding flow - of a sphere of radius \( a \) moving with velocity \( V \) in a viscous fluid with viscosity \( \eta \) and density \( \rho \). We assume, as our model, that the drag \( D \) is only dependent of \( \rho \), \( V \), \( \eta \) and \( a \). Of course, this is only true for relatively low velocities, an infinite medium, a relatively large sphere, etc., but this is our basic assumption.

Now we verify the dimensions of the parameters

\[
[D] = \text{kg m/s}^2, \quad [\rho] = \text{kg/m}^3, \quad [V] = \text{m/s,} \quad [\eta] = \text{kg/m s}, \quad [a] = \text{m}
\]  

(3.12)

and conclude that we can express \( D \) in at most \( 4 - 3 = 1 \) dimensionless constant. The common form is the Reynolds number: \( Re = \frac{\rho VL}{\eta} \). The drag may be scaled on the pressure difference between front and back of the sphere \( \frac{1}{2} \rho V^2 a^2 \), or on the viscous friction due to wall shear stress \( aV \eta \). This leads to the following functional relations

\[
D = \frac{1}{2} \rho V^2 a^2 F(Re) = aV \eta G(Re), \quad \text{where} \quad Re = \frac{\rho VL}{\eta}.
\]

(3.13)

The first one is the proper scaling for nearly inviscid flow (\( Re \) large), and the second one for very viscous flow (\( Re \) small).

Example I.9 A famous example, originally due to G.I. Taylor [2], is the analysis of the shock front propagation of a very intense (e.g. nuclear bomb) explosion. From physical considerations the radius of the shock wave front \( R \) depends, during the early stages of the explosion when the pressure inside the shock wave is much higher than outside, only on the time interval \( t \) since the explosion, the initial energy \( E \) and the initial air density \( \rho_0 \). Since \([R] = \text{m}, \quad [t] = \text{s}, \quad E = \text{kg m}^2/\text{s}^2, \quad \text{and} \quad \rho_0 = \text{kg/m}^3\), we have only \( 3 - 3 = 0 \) dimensionless groups. In other words, we can express \( R \) as

\[
R = \text{constant} \left( \frac{E}{\rho_0} \right)^{1/5} t^{2/5}.
\]

(3.14)

The full solution to the appropriate gas dynamical problem showed that the constant has a value close to unity.

Example I.10 In the convection problem

\[
\frac{\partial u}{\partial t} + U_0 \frac{\partial u}{\partial x} = 0, \quad u(x, 0) = F(x)
\]

(3.15)

there is the length scale \( U_0 t \) and a length scale, say \( L \), hidden in the initial profile \( F(x) \), as \( x \) cannot occur on its own. The dimensions of \( u \) and \( F \) are the same, say \( F_0 \), and we write \( F(x) = F_0 f(\frac{x}{L}) \).

We scale \( x = L \xi, \quad t = \frac{L}{U_0} \tau, \quad u = F_0 v, \) and \( v(\xi, 0) = f(\xi) \), to get

\[
\frac{\partial v}{\partial \tau} + \frac{\partial v}{\partial \xi} = 0,
\]

(3.16)

with solution \( v(\xi, \tau) = f(\xi - \tau) \).

Similarity solutions

If the problem contains no other length scale than the spatial variable \( x \) itself and no other time scale than the time variable \( t \) itself, dimensionless groups can only occur by combinations of \( x \) and \( t \). Thus, dimensional analysis leads naturally to similarity solutions. This is easiest explained by an example.
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Example I.11 Consider the following semi-infinite one-dimensional heat conduction problem.

\[
\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \quad \text{with} \quad T(x, 0) = 0, \quad \frac{\partial}{\partial x} T(0, t) = -Q_0
\]  

(3.17)

Scaling:

- As there is no length scale in \(x\) or \(t\), the inherent length scale can only be \(\sqrt{\alpha t}\).
- The only temperature in the problem is \(Q_0x\) or \(Q_0 \sqrt{\alpha t}\).

Therefore, we suppose:

\[
T(x, t) = Q_0 x g(\eta), \quad \text{where the similarity variable} \quad \eta = \frac{x}{\sqrt{4 \alpha t}}
\]

(3.18)

It follows that

\[
\frac{1}{2} \eta g'' + (1 + \eta^2) g' = 0
\]

(3.19)

with solution

\[
T(x, t) = Q_0 x \left[ \frac{1}{\eta \sqrt{\pi}} \exp(-\eta^2) - 1 + \text{erf}(\eta) \right]
\]

\[
= Q_0 \left[ \sqrt{\frac{4 \alpha t}{\pi}} \exp(-\eta^2) - x(1 - \text{erf}(\eta)) \right]
\]

(3.20)

where \(\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} \, d\xi\). Note that there exists no stationary solution! \(\Box\)

Example I.12 Consider the edge singularity of the (time-dependent) temperature field generated in a homogeneous and isotropic conductor by an electric field. The electric current density \(J\) and the electric field \(E\) satisfy Ohm’s law \(J = \sigma E\), where \(\sigma\) is the electric conductivity, i.e. the inverse of the specific electric resistance. For an effectively stationary current flow the conservation of electric charge leads to a vanishing divergence of the electric current density, \(\nabla \cdot J = 0\). The electric field \(E\) satisfies \(\nabla \times E = \mathbf{0}\), and therefore has a potential \(\phi\), with \(E = -\nabla \phi\), satisfying \(\nabla \cdot (\sigma \nabla \phi) = 0\). The electric conductivity \(\sigma\) is a material parameter which is quite strongly dependent on temperature. Nevertheless, to make progress we will assume a constant \(\sigma\), independent of \(T\). This, then, leads to the Laplace equation for \(\phi\)

\[
\nabla^2 \phi = 0.
\]

(3.21)

The heat dissipated as a result of the work done by the field per unit time and volume (Ohmic heating) is given by Joule’s law \(\mathbf{J} \cdot \mathbf{E}\), and leads to the heat-source distribution

\[
\sigma |\nabla \phi|^2.
\]

(3.22)

Since energy is conserved, the net rate of heat conduction and the rate of increase of internal energy are balanced by the heat source, which yields the equation for temperature \(T\)

\[
\rho C \frac{\partial T}{\partial t} = k \nabla^2 T + \sigma |\nabla \phi|^2.
\]

(3.23)

The thermal conductivity \(k\), the density \(\rho\) and the specific heat of the material \(C\) are mildly dependent on temperature, so we assumed these parameters constant.
CHAPTER I. THE ART OF MODELLING

Since we are only interested in the role of the edge, the conductor is modelled, in cylindrical \((r, \theta)\)-coordinates, as an infinite wedge-shaped two-dimensional region (without any geometrical length scale) \(0 \leq \theta \leq \nu\) with an electric field with potential

\[
\phi(x, y) = \frac{(v/\pi)}{A} r^{\pi/v} \cos(\theta \pi/v),
\]

while the temperature distribution \(T\) due to the heat generated by this source is then given by

\[
\rho C \frac{\partial T}{\partial t} = k \nabla^2 T + \sigma A^2 r^{2\pi/v - 2}
\]

with boundary and initial conditions

\[
\frac{\partial T}{\partial \theta} = 0 \text{ at } \theta = 0, \theta = \nu, \quad T(x, y, t) \equiv 0 \text{ at } t = 0.
\]

Since there are no other (point) sources in \(r = 0\), we have the additional condition of a finite field at the origin: \(0 \leq T(0, 0, t) < \infty\). Boundary conditions and the symmetric source imply that \(T\) is a function of \(t\) and \(r\) only, so that equation (3.23) reduces to

\[
\rho C \frac{\partial T}{\partial t} = k \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) + \sigma A^2 r^{2\pi/v - 2}.
\]

Owing to the homogeneous initial and boundary conditions, the infinite geometry, and the source being a monomial in \(r\), homogeneous of the order \(2\pi/v - 2\), there is no length scale in the problem other than \(\sqrt{(kt/\rho C)}\), while the temperature \(T\) can only scale on \(\frac{\sigma A^2}{k} \frac{\pi}{\nu} \left(\frac{\pi}{\nu}\right)^{-1}\). This indicates that a similarity solution is possible of the following form

\[
T(r, t) = \frac{\sigma A^2}{4k} \left(\frac{4kt}{\rho C}\right)^{\pi/v} h(X), \quad X = \frac{\rho cr^2}{4kt},
\]

where \(X\) is a similarity variable, reducing equation (3.27) to

\[
Xh'' + (1 + X)h' - (\pi/v)h = -X^{\pi/v - 1}.
\]

This equation may be recognized as an inhomogeneous confluent hypergeometric equation in \(-X\), which has the regular solution

\[
h(X) = \text{constant} \times _1F_1(-\pi/v; 1; -X) - \left(\frac{\pi}{v}\right)^2 X^{\pi/v}
\]

where \(_1F_1(a; b; z)\) is the regular confluent hypergeometric function.

From the asymptotic expansion of \(_1F_1(-\pi/v; 1; -X)\) and the initial-value condition, the integration constant is found to be \((v/\pi)\Gamma(\pi/v)\). Putting everything together, we have the solution

\[
T(x, y, t) = \frac{\sigma v^2 A^2}{4\pi^2 k} \left[ \left(\frac{4kt}{\rho C}\right)^{\pi/v} \Gamma\left(1 + \frac{\pi}{v}\right) _1F_1\left(-\frac{\pi}{v}; 1; -\frac{\rho Cr^2}{4kt}\right) r^{2\pi/v} \right].
\]

\[
\square
\]

4 Scaling of the compressible Navier-Stokes equations

One of the most fruitful uses of scaling is the hierarchy it provides in comprehensive and rich models. In most applications, such over-complete models are not truly adapted to the problem, and therefore not a true, “lean” model as we discussed above. From a suitable, inherent scaling the order of magnitude...
of the various contributions or effects can be estimated, leading to a hierarchy on which we can base an
asymptotic modelling.

We will give here a (maybe the) most important example, the scaling and reduction of the compressible
Navier-Stokes equations.

The original laws of mass, momentum and energy conservation, written in terms of pressure \(p\), density
\(\rho\), velocity vector \(\mathbf{v}\), viscous stress tensor \(\mathbf{\tau}\), internal energy, and heat flux vector \(\mathbf{q}\), are given by

\[
\begin{align*}
\text{mass:} & \quad \frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (4.1a) \\
\text{momentum:} & \quad \frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{vv}) = -\nabla p + \nabla \cdot \mathbf{\tau} \quad (4.1b) \\
\text{energy:} & \quad \frac{\partial}{\partial t} (\rho e + \frac{1}{2} \rho \mathbf{v}^2) + \nabla \cdot (\rho e \mathbf{v} + \frac{1}{2} \rho \mathbf{v}^2 \mathbf{v}) = -\nabla \cdot \mathbf{q} - \nabla \cdot (p \mathbf{v}) + \nabla \cdot (\nabla \cdot \mathbf{\tau}) \quad (4.1c)
\end{align*}
\]

Depending of the application, it is often convenient to introduce enthalpy \(h = e + \rho / p\), or entropy \(s\) and
temperature \(T\) via the fundamental law of thermodynamics for a reversible process

\[T \, ds = de + p \, d\rho^{-1} = dh - \rho^{-1} \, dp. \quad (4.2)\]

With \(d/dt = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\) for the convective derivative, the above conservation laws may be reduced to

\[
\begin{align*}
\text{mass:} & \quad \frac{\partial}{\partial t} \rho = -\rho \nabla \cdot \mathbf{v} \quad (4.3a) \\
\text{momentum:} & \quad \rho \frac{\partial}{\partial t} \mathbf{v} = -\nabla p + \nabla \cdot \mathbf{\tau} \quad (4.3b) \\
\text{energy:} & \quad \rho \frac{\partial}{\partial t} e = -\nabla \cdot \mathbf{q} - p \nabla \cdot \mathbf{v} + \mathbf{\tau} : \nabla \mathbf{v} \quad (4.3c) \\
\rho \frac{\partial}{\partial t} h & = \frac{\partial}{\partial t} p - \nabla \cdot \mathbf{q} + \mathbf{\tau} : \nabla \mathbf{v} \quad (4.3d) \\
\rho T \frac{\partial}{\partial t} s & = -\nabla \cdot \mathbf{q} + \mathbf{\tau} : \nabla \mathbf{v} \quad (4.3e)
\end{align*}
\]

For an ideal gas we have \(de = C_V dT\) and \(dh = C_p dT\) such that

\[
\begin{align*}
\text{energy 1:} & \quad \rho C_V \frac{\partial}{\partial t} T = -\nabla \cdot \mathbf{q} - p \nabla \cdot \mathbf{v} + \mathbf{\tau} : \nabla \mathbf{v} \quad (4.3f) \\
\text{energy 2:} & \quad \rho C_p \frac{\partial}{\partial t} T = \frac{\partial}{\partial t} p - \nabla \cdot \mathbf{q} + \mathbf{\tau} : \nabla \mathbf{v} \quad (4.3g)
\end{align*}
\]

For an incompressible fluid we have \(T \, ds = C_p dT\) with

\[
\begin{align*}
\text{energy 3:} & \quad \rho C_p \frac{\partial}{\partial t} T = -\nabla \cdot \mathbf{q} + \mathbf{\tau} : \nabla \mathbf{v} \quad (4.3h)
\end{align*}
\]

\(C_V\) is the heat capacity or specific heat at constant volume. \(C_p\) is the heat capacity or specific heat at
constant pressure [18]. In general (for an ideal gas) they may depend on temperature. For a perfect gas
they are constants. For an incompressible fluid like a liquid \(C_V\) is practically equal to \(C_p\).

**Constitutive equations**

These 5 equations are less than the number of unknowns. Therefore, the mathematical description is not
complete without some “closure” relations, describing properties of the matter and certain instantaneous
(both in time and space) interactions between material parameters. These relations are called the constitutive
equations. They are not as basic as the conservation laws, and their form will depend on the model
adopted. In fact, this part of the model, the constitutive equations, may be called: the physical model.
We will consider here an ideal, heat conducting and viscous gas.

\[
\begin{align*}
\text{Ideal gas relation:} & \quad p = \rho R T, \quad (4.4) \\
\text{Fourier’s heat flux model:} & \quad \mathbf{q} = -k \nabla T, \quad (4.5) \\
\text{Newton’s viscous stress tensor:} & \quad \mathbf{\tau} = \eta (\nabla \mathbf{v} + (\nabla \mathbf{v})^\top) - \frac{2}{3} \eta (\nabla \cdot \mathbf{v}) \mathbf{I}. \quad (4.6)
\end{align*}
\]
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Scaling, non-dimensionalisation:

Assume that we have the following typical scales for velocity, pressure, density and temperature.

\[ \nu \text{ with } \nu_0, \Delta \nu; \quad p \text{ with } p_0, \Delta p; \quad \rho \text{ with } \rho_0, \Delta \rho; \quad T \text{ with } \Delta T; \quad x \text{ with } L; \quad t \text{ with } f^{-1} \]

where a subscript “\(0\)” refers to the primary variable, and \(\Delta\) denotes a typical difference. The typical time is written as the inverse of frequency \(f\). Note that sometimes we have more candidates for a scaling parameter. For example, the typical frequency may be enforced by an external source or instability, or may be inherited from the intrinsic hydrodynamic frequency \(\nu_0/L\), or the inverse diffusion time \(\kappa/\rho_0C_PL^2\).

The scaled equations are now, symbolically, with energy equation 2,

\[
\begin{align*}
  f \Delta \rho, \quad \frac{\nu_0 \Delta \rho}{L} &= \frac{\rho_0 \Delta \nu}{L} \\
  f \rho_0 \Delta v, \quad \frac{\rho_0 \nu_0 \Delta v}{L} &= \frac{\Delta p}{L} \cdot \frac{\eta \Delta v}{L^2} \\
  f \rho_0 C_P \Delta T, \quad \frac{\rho_0 C_P \nu_0 \Delta T}{L} &= f \Delta p, \quad \frac{\nu_0 \Delta p}{L}, \quad \frac{\kappa \Delta T}{L^2}, \quad \frac{\eta \Delta v^2}{L^2}
\end{align*}
\]

From gas laws we know that the typical sound speed squared \(c_0^2\) scales on both \(p_0/\rho_0\) as \(\Delta p/\Delta \rho\) (not exactly, only in order of magnitude). Furthermore, we will take here the common situation where \(\Delta v = \Theta(\nu_0)\), and \(\kappa\) and \(\eta\) are constants. With a rescaling such that all coefficients become dimensionless, we have

\[
\frac{fL}{\nu_0} \Delta p, \quad \frac{\Delta p}{\rho_0 c_0^2} = 1
\]

\[
\frac{fL}{\nu_0}, \quad 1 = \frac{\Delta p}{\rho_0 \nu_0}, \quad \frac{\eta}{\rho_0 \nu_0 L}
\]

\[
\frac{fL}{\nu_0}, \quad 1 = \frac{fL \Delta p}{\rho_0 C_P \nu_0 \Delta T}, \quad \frac{\Delta p}{\rho_0 C_P \Delta T}, \quad \frac{\kappa}{\rho_0 C_P \nu_0 L}, \quad \frac{\nu_0 \eta}{\rho_0 C_P L \Delta T}
\]

4.1 Some dimensionless groups with their common names:

- **Reynolds**: \(Re = \frac{\rho_0 \nu_0 L}{\eta}\)
- **Prandtl**: \(Pr = \frac{C_P \eta}{\kappa}\)
- **Mach**: \(M = \frac{\nu_0}{c_0}\)
- **Strouhal**: \(Sr = \frac{fL}{\nu_0}\)
- **Fourier**: \(Fo = \frac{\kappa f^{-1}}{\rho_0 C_P L^2}\)
- **Euler**: \(Eu = \frac{\Delta p}{\rho_0 \nu_0^2}\)
- **Helmholtz**: \(He = \frac{fL}{\kappa}\)
- **Eckert**: \(Ec = \frac{\nu_0^2}{C_P \Delta T}\)
- **Peclet**: \(Pe = \frac{\rho_0 C_P \nu_0 L}{\kappa}\)

The Prandtl number \(Pr\) depends only of the material, and is for most gases and fluids of order 1. Many dimensionless number are related. For example, \(Pe = Pr Re\), \(He = Sr M\), and \(Sr Fo Pr Re = 1\).

A Fourier number \(Fo = 1\) corresponds to a typical time, determined by the heat diffusion process. A Strouhal number \(Sr = 1\) corresponds to a scaling frequency \(f = \nu_0/L\), corresponding to hydrodynamical convection processes. An Euler number \(Eu = 1\) corresponds to a pressure scaled on hydrodynamic pressure variations, \(\rho_0 \nu_0^2\), while the combination \(Eu M^2 = 1\) corresponds to a pressure scaled on pressure variations.
4. SCALING OF THE COMPRESSIBLE NAVIER-STOKES EQUATIONS

variations due to compressible effects, \( \rho_0 c_0^2 \). If the reference speed is taken equal to the sound speed, we have \( M = 1 \).

The scaled equations are now (written in dimensionless variables)

\[
Eu M^2 \left( Sr \frac{\partial}{\partial t} \rho + \mathbf{v} \cdot \nabla \rho \right) = -\rho \nabla \cdot \mathbf{v} \tag{4.7a}
\]

\[
Sr \rho \frac{\partial}{\partial t} \mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -Eu \nabla p + \frac{1}{Re} \nabla \cdot \mathbf{\tau} \tag{4.7b}
\]

\[
Sr \rho \frac{\partial}{\partial t} T + \rho \mathbf{v} \cdot \nabla T = Eu Ec \left( Sr \frac{\partial}{\partial t} p + \mathbf{v} \cdot \nabla p \right) + \frac{1}{Pe} \nabla^2 T + \frac{Ec}{Re} \mathbf{\tau} : \nabla \mathbf{v} \tag{4.7c}
\]

4.2 Asymptotic reductions of the Navier-Stokes equations

When \( M, Re, Sr, Ec \), etc. become small or large, we may derive from equations (4.7) various reduced models by assuming the respective terms to vanish or dominate completely, for example:

- incompressible: \( \nabla \cdot \mathbf{v} = 0 \),
- inviscid: \( \rho_0 \left( \frac{\partial}{\partial t} \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} \right) + \nabla p = 0 \),
- highly viscous: \( \nabla p = \eta \nabla^2 \mathbf{v} \),
- sound waves: \( \frac{\partial}{\partial t} \rho + \rho_0 \nabla \cdot \mathbf{v} = 0 \) \( \frac{\partial}{\partial t} \rho + \rho_0 \nabla p = 0 \) \( \rightarrow \frac{\partial}{\partial t} \rho + \rho_0 c_0^2 \nabla^2 p = 0 \),
- convection-diffusion: \( \rho C_P \left( \frac{\partial}{\partial t} T + \mathbf{v} \cdot \nabla T \right) = \kappa \nabla^2 T \).

Potential flow

Another important reduction, not immediately obtainable from small parameter considerations, is the scalar velocity potential that may be introduced for irrotational flow:

if \( \nabla \times \mathbf{v} = 0 \), there exists a velocity potential \( \phi \) with \( \mathbf{v} = \nabla \phi \).

In incompressible flow this potential is independent of pressure (except indirectly via boundary conditions) and satisfies Laplace’s equation \( \nabla^2 \phi = 0 \).

Bernoulli’s law

In incompressible inviscid flow the momentum equation can be integrated along a streamline, leading to an equation, known as Bernoulli’s law, that describes conservation of mechanical energy density:

\[
\mathbf{v} \cdot \nabla \mathbf{v} = \frac{1}{2} \nabla |\mathbf{v}|^2 + (\nabla \times \mathbf{v}) \times \mathbf{v},
\]

integrate along streamline, noting that \( [(\nabla \times \mathbf{v}) \times \mathbf{v}] \cdot ds = 0 \):

\[
0 = \int \left[ \frac{1}{2} \rho_0 |\mathbf{v}|^2 + \rho_0 (\nabla \times \mathbf{v}) \times \mathbf{v} + \nabla p \right] \cdot ds = \frac{1}{2} \rho_0 |\mathbf{v}|^2 + p - \text{constant}.
\]
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Exercises

1. The drag $D$ of a moving ship, due to viscous effects and wave generation, depends on its length $L$, velocity $V$, water viscosity $\eta$, gravity $g$ and water density $\rho$. The dimensional units are $[D] = \text{kg m/s}^2$, $[L] = \text{m}$, $[V] = \text{m/s}$, $[\eta] = \text{kg/ms}$, $[g]=\text{m/s}^2$, and $[\rho]=\text{kg/m}^3$. By how many dimensionless groups is the problem completely described? Give an example of such a set of dimensionless groups (these are not uniquely defined).

2. Analyse example I.12 in terms of Buckingham’s theorem. Verify the dimensional groups, including their number.

3. Determine the small or large dimensionless numbers corresponding to the above reduced models (4.8). Be aware of possible combined limits.

4. Reconsider example I.1. Make the problem complete by adding boundary and initial conditions. Make the problem dimensionless by scaling on the inherent time and length scales. Determine conditions, in terms of a dimensionless number, for which the diffusion term can be neglected. Note that the order of the differential equation is then reduced from 2 to 1. What are the consequences for the boundary and/or initial conditions?

5. Simplify, by suitable scaling of the independent variables, the equations of examples ?? (i) and (iii) such that the coefficients become just equal to 1.
4. SCALING OF THE COMPRESSIBLE NAVIER-STOKES EQUATIONS
Chapter II

Perturbation methods

Inherent in any modelling is the hierarchy in importance of the various effects that constitute the model. Therefore, certain effects in any modelling will be small. Sometimes small but not small enough to be ignored, and sometimes small but in a non-uniform way such that they are important locally.

For an efficient solution, and to obtain qualitative insight, it make sense to utilize this “smallness”. Methods that systematically exploit such inherent smallness are called “perturbation methods”.

Perturbation methods have a long history. Before the time of the numerical methods and the computer, perturbation methods were the only way to increase the applicability of available exact solutions to difficult, otherwise intractable problems. Nowadays, perturbation methods have their use as a natural step in the process of systematic modelling, the insight it provides in the nature of singularities occurring in the problem and typical parameter dependencies, and sometimes the speed of practical calculations.

1 Introduction

We have seen above that a real-world problem can be described by a hierarchy of models, such that a higher level model is more comprehensive and more accurate than one from a lower level. Now suppose that we have a fairly good model, describing the dominating phenomena in good order of magnitude. And suppose that we are interested in improving on this model by adding some previously ignored aspects or effects. In general, this implies a very abrupt change in our model. The equations are more complex and more difficult to solve.

So it seems that with our model, necessarily also our solution method is abruptly more complex. Sometimes, this is an argument to retain the simpler model. This is, however, only true if we are merely interested in exact or numerically “exact” solutions. We have to keep in mind that an exact solution of an approximate model is not better than an approximate solution of an exact model. So there is absolutely no reason to demand the solution to be more exact than the corresponding model.

Let’s go back to our “fairly good”, improved model. The effects we added are relatively small; otherwise, the previous lower level model was not fairly good as we assumed, but just completely wrong. Usually, this smallness is quantified by small dimensionless parameters occurring in the equations and (or) boundary conditions. This is a generic situation. The transition between a higher-level and a lower-level theory
1. INTRODUCTION

is characterized by the presence of one or more inherent modelling parameters, which are large or small, and yield in the limit a simpler description.

Examples are: simplified geometries reducing the spatial dimension, small amplitudes allowing linearization, low velocities and long time scales in flow problems allowing incompressible description, small relative viscosity allowing inviscid models, zero or infinite lengths rather than finite lengths, etc.

If we accept approximate solutions, where the approximation is based on the inherent small or large modelling parameters, we do have the possibilities to gradually increase the complexity of a model, and study small but significant effects in the most efficient way.

The methods utilizing systematically this approach are called “perturbations methods”. The approximation constructed is almost always an asymptotic approximation, i.e. where the error reduces with the small or large parameter.

Usually, a distinction is made between regular and singular perturbations. A (loose definition of a) regular perturbation problem is where the approximate problem is everywhere close to the unperturbed problem. This, however, depends of course on the domain of interest and, as we will see, on the choice of coordinates.

If a problem is regular without any need for other than trivial reformulations, the construction of an asymptotic solution is straightforward. In fact, it forms the usual strategy in modelling when terms are linearised or effects are neglected. The more interesting perturbation problems are those where this straightforward approach fails.

We will consider here four methods relevant in the presented modelling problems. The first two are examples of regular perturbation methods, but only after a suitable coordinate transformation. They are called the method of slow variation, where the typical axial length scale is much greater than the transverse length scale. The next one is the Lindstedt-Poincaré method or the method of strained coordinates, for periodic processes. Here, the intrinsic timescale (~ the period of the solution) is unknown and has to be found.

The other methods are of singular perturbation type, because there is no coordinate transformation possible that renders the problem into one of regular type. The first one is the method of multiple scales and may be considered as a combination of both methods above, as now several (long, short, shorter) time scales act together. This cannot be repaired by a single coordinate transformation. Therefore, the problem is temporarily reformulated into a higher dimensional problem by taking the various length scales apart. Then the problem is regular again, and can be solved. A refinement of this method is the WKB method, where the coordinate transformation of the fast variable becomes itself slowly varying. The last singular perturbation method considered here is the method of matched asymptotic expansions. Also here different scalings are necessary, but they don’t exist together, but in spatially distinct regions (boundary layers).

Before we can introduce the methods, we have to define our terminology of asymptotic approximations and asymptotic expansions.
2 Asymptotic approximations and expansions

2.1 Asymptotic approximations

In order to describe qualitatively the way a function \( f \) of parameter \( \varepsilon \) behaves near a point of interest, say \( \varepsilon = 0 \) (equivalent to any other value by a simple translation), we have the so-called order symbols \( \mathcal{O}, o, \) and \( \mathcal{O}_s \). Often \( \varepsilon = 0 \) is the lower limit of a parameter range, and we have the tacit assumption that \( \varepsilon \downarrow 0 \).

Definition II.1.

1. \( f(\varepsilon) = \mathcal{O}(\phi(\varepsilon)) \) as \( \varepsilon \to 0 \) if there are constants \( K \) and \( \varepsilon_1 \) (independent of \( \varepsilon \)) such that
   \[
   |f(\varepsilon)| \leq K|\phi(\varepsilon)| \quad \text{for} \quad 0 < \varepsilon < \varepsilon_1.
   \]

2. \( f(\varepsilon) = o(\phi(\varepsilon)) \) as \( \varepsilon \to 0 \) if for every \( \delta > 0 \), there is an \( \varepsilon_1 \) (independent of \( \varepsilon \)) such that
   \[
   |f(\varepsilon)| \leq \delta|\phi(\varepsilon)| \quad \text{for} \quad 0 < \varepsilon < \varepsilon_1.
   \]

3. \( f(\varepsilon) = \mathcal{O}_s(\phi(\varepsilon)) \) as \( \varepsilon \to 0 \) if \( f(\varepsilon) = \mathcal{O}(\phi(\varepsilon)) \) and \( f(\varepsilon) \neq o(\phi(\varepsilon)) \).

Theorem II.2.

1. If \( f = o(\phi) \) as \( \varepsilon \to 0 \), then also \( f = \mathcal{O}(\phi) \).

2. If the limit \( \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\phi(\varepsilon)} \) exists as a finite number, then \( f = \mathcal{O}(\phi) \) as \( \varepsilon \to 0 \).

3. If the limit \( \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\phi(\varepsilon)} \) exists as a finite number \( \neq 0 \), then \( f = \mathcal{O}_s(\phi) \) as \( \varepsilon \to 0 \).

4. If \( \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\phi(\varepsilon)} = 0 \), then \( f = o(\phi) \) as \( \varepsilon \to 0 \).

Example II.3

\[
\begin{align*}
\varepsilon \sin(\varepsilon) &= \mathcal{O}_s(\varepsilon^2), \quad \varepsilon \to 0, \\
\varepsilon \cos(\varepsilon) &= \mathcal{O}(1), \quad \varepsilon \to 0, \\
\varepsilon^n &= o(1), \quad \varepsilon \to 0, \text{ for any positive } n, \\
e^{-1/\varepsilon} &= o(\varepsilon^n), \quad \varepsilon \to 0, \text{ for any positive } n.
\end{align*}
\]

\( e^{-1/\varepsilon} \) is called a transcendentally (TST) or exponentially small term (EST) and can be ignored asymptotically against any power of \( \varepsilon \).

Definition II.4. \( \phi(\varepsilon) \) is an asymptotic approximation to \( f(\varepsilon) \) as \( \varepsilon \to 0 \) if

\[
f(\varepsilon) = \phi(\varepsilon) + o(\phi) \quad \text{as} \quad \varepsilon \to 0,
\]

sometimes more compactly denoted by \( f \sim \phi \).
2. ASYMPTOTIC APPROXIMATIONS AND EXPANSIONS

If \( f \) and \( \phi \) depend on \( x \), this definition remains valid pointwise, i.e. for \( x \) fixed. It is, however, useful to extend the definition to uniformly valid approximations.

**Definition II.5.** If \( f(x; \varepsilon) \) and \( \phi(x; \varepsilon) \) are continuous functions for \( x \in \mathcal{D} \) and \( 0 < \varepsilon < a \), we call \( \phi(x; \varepsilon) \) a uniformly valid asymptotic approximation to \( f(x; \varepsilon) \) for \( x \in \mathcal{D} \) as \( \varepsilon \to 0 \) if for any positive number \( \delta \) there is an \( \varepsilon_1 \) (independent of \( x \) and \( \varepsilon \)) such that
\[
|f(x; \varepsilon) - \phi(x; \varepsilon)| \leq \delta |\phi(x; \varepsilon)| \quad \text{for } x \in \mathcal{D} \text{ and } 0 < \varepsilon < \varepsilon_1.
\]

We write: \( f(x; \varepsilon) = \phi(x; \varepsilon) + o(\phi) \) uniformly in \( x \in \mathcal{D} \) as \( \varepsilon \to 0 \). Note that \( \mathcal{D} \) may depend on \( \varepsilon \).

**Example II.6** Consider \( \mathcal{D} = [0, 1], \; 0 < \varepsilon < 1 \). Then we have \( \cos(\varepsilon x) = 1 + o(1) \) as \( \varepsilon \to 0 \) uniformly in \( \mathcal{D} \), since for any given \( \delta \) we can choose \( \varepsilon_1 = \sqrt{\delta} \), such that
\[
|\cos(\varepsilon x) - 1| \leq \varepsilon^2 x^2 \leq \varepsilon^2\varepsilon_1^2 = \delta.
\]

**Example II.7** Although \( \cos(x/\varepsilon) = O(1) \) uniformly in \( x \in [0, 1] \) for \( \varepsilon \to 0 \), there is no constant \( K \) such that \( \cos(x/\varepsilon) = K + o(1) \).

**Example II.8** \( x + \sin(\varepsilon x) + e^{-x/\varepsilon} = x + \varepsilon x + O(\varepsilon^3) \) as \( \varepsilon \to 0 \) for all \( x \neq 0 \), but not uniformly in \( x \in [0, 1] \). If \( x = 0 \), the otherwise exponentially small term is not small anymore and equal to unity.

In the figure we may compare the original \( x + \sin(\varepsilon x) + e^{-x/\varepsilon} \) and its non-uniform asymptotic approximation \( x + \varepsilon x \) for \( \varepsilon = 0.01 \). Note that difference between the original function and its non-uniform asymptotic approximation is typically large in a neighbourhood of \( x = 0 \), while the size of this neighbourhood is \( x = O(\varepsilon) \). This neighbourhood is an example of a boundary layer. The occurrence and behaviour of boundary layers will be discussed in more detail in the next section.

2.2 Asymptotic expansions

**Definition II.9.** The sequence \( \{\mu_n(\varepsilon)\}_{n=0}^{\infty} \) is called an asymptotic sequence, if \( \mu_{n+1}(\varepsilon) = o(\mu_n(\varepsilon)) \), as \( \varepsilon \to 0 \), for each \( n = 0, 1, 2, \ldots \).

**Example II.10** Examples of asymptotic sequences (as \( \varepsilon \to 0 \)) are
\[
\begin{align*}
\mu_n(\varepsilon) &= \varepsilon^n, \quad \mu_n(\varepsilon) = \varepsilon^{\frac{1}{2^n}}, \quad \mu_n(\varepsilon) = \tan^n(\varepsilon), \quad \mu_n(\varepsilon) = \ln(\varepsilon)^{-n}, \\
\mu_n(\varepsilon) &= \varepsilon^p \ln(e)^q \text{ where } p = 0, 1, 2, \ldots, \quad q = 0, \ldots, p \text{ and } n = \frac{1}{p}(p+3) - q.
\end{align*}
\]
CHAPTER II. PERTURBATION METHODS

Definition II.11. If \( \{\mu_n(\varepsilon)\}_{n=0}^{\infty} \) is an asymptotic sequence, then \( f(\varepsilon) \) has an asymptotic expansion of \( N \) terms with respect to this sequence, given by

\[
f(\varepsilon) \sim \sum_{n=0}^{N-1} a_n \mu_n(\varepsilon)
\]

where \( a_n \) are independent of \( \varepsilon \), if

\[
f(\varepsilon) - \sum_{n=0}^{M} a_n \mu_n(\varepsilon) = o(\mu_M) \quad \text{as} \quad \varepsilon \to 0
\]

for each \( M = 0, \ldots, N - 1 \). \( \mu_n(\varepsilon) \) is called a gauge-function. If \( \mu_n(\varepsilon) = \varepsilon^n \), we call the expansion an asymptotic power series.

Definition II.12. Two functions \( f \) and \( g \) are asymptotically equal up to \( n \) terms, with respect to the asymptotic sequence \( \{\mu_n\} \), if \( f - g = o(\mu_n) \). If the remaining error is clear from the context, this is sometimes denoted as \( f \sim g \).

Asymptotic expansions based on the same gauge functions may be added. They may be multiplied if the products of the gauge functions can be asymptotically ordered.

In contrast to ordinary series expansions, defined for an infinite number of terms, we consider in asymptotic expansions only a finite (\( N \)) number of terms. For \( N \to \infty \) the series may either converge or diverge, but this is irrelevant for the asymptotic behaviour. In addition it may be worthwhile to note that it is not necessary for a convergent asymptotic expansion to converge to the expanded function.

For given \( \{\mu_n(\varepsilon)\}_{n=0}^{\infty} \), \( a_n \) can be determined uniquely by the following recursive procedure (provided \( \mu_n \) are nonzero for \( \varepsilon \) near 0 and each of the limits below exist)

\[
a_0 = \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\mu_0(\varepsilon)},
\]

\[
a_1 = \lim_{\varepsilon \to 0} \frac{f(\varepsilon) - a_0 \mu_0(\varepsilon)}{\mu_1(\varepsilon)},
\]

\[
\vdots
\]

\[
a_{N-1} = \lim_{\varepsilon \to 0} \frac{f(\varepsilon) - \sum_{n=0}^{N-2} a_n \mu_n(\varepsilon)}{\mu_{N-1}(\varepsilon)}.
\]

Example II.13 Given different asymptotic sequences, a function may have different asymptotic expansions.

\[
\tan(\varepsilon) = \varepsilon + \frac{1}{3} \varepsilon^3 + \frac{2}{15} \varepsilon^5 + O(\varepsilon^7)
\]

\[
= \sin(\varepsilon) + \frac{1}{2} \sin^3(\varepsilon) + \frac{3}{8} \sin^5(\varepsilon) + O(\sin^7(\varepsilon))
\]

\[
= \varepsilon \cos(\varepsilon) + \frac{3}{8} (\varepsilon \cos(\varepsilon))^3 + \frac{111}{256} (\varepsilon \cos(\varepsilon))^5 + O((\varepsilon \cos(\varepsilon))^7).
\]

Example II.14 The asymptotic expansion \( \sum_{n=1}^{N} n! c_n \varepsilon^n \) diverges as \( N \to \infty \) if \( \varepsilon \neq 0 \).
2. ASYMPTOTIC APPROXIMATIONS AND EXPANSIONS

Example II.15 Different functions may have the same asymptotic expansion.

\[ \cos(e) = 1 - \frac{1}{2}e^2 + \frac{1}{24}e^4 + O(e^6). \]
\[ \cos(e) + e^{-1/e} = 1 - \frac{1}{2}e^2 + \frac{1}{24}e^4 + O(e^6). \]

Note that both asymptotic expansions, considered as regular power series in \( e \), converge to \( \cos(e) \) rather than to \( \cos(e) + e^{-1/e} \).

Theorem II.16. An asymptotic expansion vanishes only if the coefficients vanish, i.e.

\[ \{ \mu_0(e) a_0 + \mu_1(e) a_1 + \mu_2(e) a_2 + \ldots = 0 \quad (e \to 0) \} \iff \{ a_0 = a_1 = a_2 = \ldots = 0 \}. \]

Proof: \( \{ \mu_n \} \) are asymptotically ordered, so both \( \mu_0 a_0 = -\mu_1 a_1 - \ldots = O(\mu_1) \) and \( \mu_1 = o(\mu_0) \). So there is a positive constant \( K \) such that for any positive \( \delta \) there is an \( e \)-interval where \( |\mu_0 a_0| < \delta K|\mu_0| \), which is only possible if \( a_0 = 0 \). This may now be repeated for \( a_1 \), etc.

2.3 Perturbation problems

The assumed existence of an asymptotic expansions yields a class of methods to solve otherwise intractable problems depending on a typically small parameter. Such methods are called perturbation methods.

If \( a(e) \) is implicitly given as the solution of an algebraic equation

\[ F(a; e) = 0 \] (2.2)

and both \( a(e) \) and \( F(a; e) \) have an asymptotic series expansion with the same gauge functions, \( a(e) \) may be determined asymptotically by the following perturbation method. We expand \( a \), substitute this expansion in \( F \), and expand \( F \) to obtain

\[ a(e) = \mu_0(e) a_0 + \mu_1(e) a_1 + \ldots, \] (2.3a)
\[ F(a; e) = \mu_0(e) F_0(a_0) + \mu_1(e) F_1(a_1, a_0) + \mu_2(e) F_2(a_2, a_1, a_0) + \ldots = 0. \] (2.3b)

From theorem II.16 it follows that each term \( F_n \) vanishes, and the sequence of coefficients \( (a_n) \) can be determined by induction:

\[ F_0(a_0) = 0, \quad F_1(a_1, a_0) = 0, \quad F_2(a_2, a_1, a_0) = 0, \quad \text{etc.} \] (2.4)

It should be noted that finding the sequence of gauge functions \( (\mu_n) \) is of particular importance. This done iteratively. First the order of magnitude of \( a \) should be determined by seeking the asymptotic scaling \( a(e) = \gamma(e) A(e) \) which yields in the limit \( e \to 0 \) a meaningful \( A = O(1) \). This is called a distinguished limit, while the reduced equation for \( A(0) \) is called a significant degeneration. (There may be more than one.) The first term of the gauge functions that occurs is now \( \mu_0(e) = \gamma(e) \), while \( a_0 = A(0) \). Now the procedure may be repeated for the new unknown \( a(e) - \mu_0(e) a_0 \), and so on. Usually, the rest of the sequence \( (\mu_n) \) can be guessed from the structure of the defining equation \( F = 0 \).

We illustrate this procedure by the following example.
Example II.17 Consider the zeros for $\varepsilon \to 0$ of the polynomial

$$x^3 - \varepsilon x^2 + 2\varepsilon^3 x + 2\varepsilon^6 = 0.$$ 

From the structure of the problem it seems reasonable to assume that the solutions $x^{(1)}, x^{(2)}, x^{(3)}$ have an asymptotic expansion in powers of $\varepsilon$. However, not clear is the order of magnitude of the leading order term:

$$x(\varepsilon) = \varepsilon^n (X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \mathcal{O}(\varepsilon^3))$$

Therefore, we have to determine $n$ first. This is done by balancing the terms. We scale

$$x = \varepsilon^n X(\varepsilon), \quad X = \mathcal{O}(1)$$

and we seek those $n$ that produce under the limit $\varepsilon \to 0$ a non-trivial limit. We compare asymptotically the coefficients in the equation that remain after scaling:

$$\varepsilon^{3n} X^3 - \varepsilon^{1+2n} X^2 + 2\varepsilon^{3+n} X + 2\varepsilon^6 = 0.$$ 

In order to have a meaningful (or “significant”) degenerate solution $X(0) = \mathcal{O}(1)$ at least two terms of the equation should be asymptotically equivalent, and at the same time of leading order when $\varepsilon \to 0$. So this leaves us with the task to compare as a function of $n$ the powers $3n, 1 + 2n, 3 + n, 6$.

Consider the figure II.1 The drawn lines denote the logarithm of the powers of $\varepsilon$, that occur in the coefficients of the equation considered. At the crossings of these lines, denoted by the open and closed circles, we find the candidates of distinguished limits, i.e. the points where at least two coefficients are asymptotically equivalent. Finally, only the closed circles are the distinguished limits, because these are located along the lower envelope (thick broken line) and therefore correspond to leading order terms when $\varepsilon \to 0$.

We have now three cases.

$n = 1$.

$$\varepsilon^3 X^3 - \varepsilon^3 X^2 + 2\varepsilon^4 X + 2\varepsilon^6 = 0, \quad \text{or} \quad X^3 - X^2 + 2\varepsilon X + 2\varepsilon^3 = 0.$$
2. ASYMPTOTIC APPROXIMATIONS AND EXPANSIONS

If we assume the expansion \( X = X_0 + \varepsilon X_1 + \ldots \), we have finally
\[
X_3^0 - X_0^2 = 0, \quad 3X_0^0X_1 - 2X_0X_1 + 2X_0 = 0, \quad \text{etc.}
\]
and so \( X_0 = 1 \), and \( X_1 = -2 \), etc.. Note: \( \text{not } X_0 = 0 \) because that would change the order of the scaling!

\[
\varepsilon^6 X^3 - \varepsilon^5 X^2 + 2\varepsilon^6 X + 2\varepsilon = 0, \quad \text{or} \quad \varepsilon X^3 - X^2 + 2X + 2\varepsilon = 0.
\]

If we assume the expansion \( X = X_0 + \varepsilon X_1 + \ldots \), we have finally
\[
-X_0^2 + 2X_0 = 0, \quad \text{etc.}
\]
and so \( X_0 = 2 \), etc..

\[
\varepsilon^9 X^3 - \varepsilon^7 X^2 + 2\varepsilon^6 X + 2\varepsilon = 0, \quad \text{or} \quad \varepsilon X^3 - \varepsilon X^2 + 2X + 2 = 0.
\]

If we assume the expansion \( X = X_0 + \varepsilon X_1 + \ldots \), we have finally
\[
2X_0 + 2 = 0, \quad \text{etc.}
\]
and so \( X_0 = -1 \), etc..

Not always is it so easy to guess the general form of the gauge functions. Then all terms have to be estimated iteratively, by a similar process of balancing as for the leading order term. See exercise 2.

An asymptotic expansion of a function of variable \( x \) and small parameter \( \varepsilon \) of the following form
\[
f(x; \varepsilon) = \sum_{n=0}^{N-1} a_n(x; \varepsilon)\mu_n(\varepsilon) + \mathcal{O}(\mu_N), \quad a_n = \mathcal{O}(1),
\]
appears to be too general to be of practical use. The restriction that \( a_n \) depend of \( x \) only appears to be fruitfull. This is called

2.4 Asymptotic expansions of Poincaré type

Definition II.18. If \( \{\mu_n(\varepsilon)\}_{n=0}^{\infty} \) is an asymptotic sequence, and \( f(x; \varepsilon) \) has an asymptotic expansion of \( N \) terms with respect to this sequence, given by
\[
f(x; \varepsilon) \sim \sum_{n=0}^{N-1} a_n(x)\mu_n(\varepsilon)
\]
where shape functions \( a_n(x) \) are independent of \( \varepsilon \), this expansion is called a Poincaré expansion.

Definition II.19. If a Poincaré expansion is uniform in \( x \) on a given domain \( D \) (the \( \varepsilon \)-intervals only depend on the \( \delta \)'s, not on \( x \)) this expansion is called a regular expansion. If not, the expansion is called a singular expansion\(^1\).

\(^1\)There is no uniformity in the literature on the definition of regular and singular expansions.
Regular expansions may be differentiated to the independent variable $x$.

**Example II.20** The following asymptotic power series expansion with $\mathcal{D} = \mathbb{R}$ and the gauge functions given by $\varepsilon^n$

$$
\cos(x + \varepsilon) = \cos(x) - \varepsilon \sin(x) - \frac{1}{2}\varepsilon^2 \cos(x) + \frac{1}{6}\varepsilon^3 \sin(x) + \mathcal{O}(\varepsilon^4).
$$

is uniform since $\cos(x)$ and $\sin(x)$ are bounded for all $x \in \mathbb{R}$. It follows that it is a regular expansion.

**Example II.21** The following expansion

$$
\cos(x + \varepsilon) e^{-1/\varepsilon} + \sin(x + \varepsilon) = \sin x + \varepsilon \cos x + \mathcal{O}(\varepsilon^2)
$$

is a uniform, and therefore regular expansion on any interval $[A, \infty)$, where $A > 0$. However, it is a non-uniform, and therefore singular expansion on $[0, \infty)$. In fact, on any interval $[A\varepsilon^n, \infty)$ it is regular if $\alpha < 1$, and singular if $\alpha \geq 1$.

### The role of the independent coordinate

It is absolutely vital for the appreciation of the perturbations methods to be considered below, to note the role of the choice of the independent variable $x$ in a Poincaré expansion. By suitable linear coordinate transformations of the type $x = \lambda(\varepsilon) + \delta(\varepsilon) \xi$ we can change and optimize the domain of uniformity, and filter out asymptotically specific behaviour that belongs to one length scale. This filter property is especially useful when asymptotically analysing models.

**Example II.22**

- $\sin(x + \varepsilon x + \varepsilon^2 x) = \sin(x) + \varepsilon x \cos(x) + \varepsilon^2 (x \cos x - \frac{1}{2}x^2 \sin x) + \mathcal{O}(\varepsilon^3)$, which is only uniform on an interval $[0, A]$, but if we introduce $\xi = (1 + \varepsilon)x$, we have $\sin(\xi + \varepsilon^2 \xi) = \sin(\xi) + \mathcal{O}(\varepsilon^3)$ uniform in $x \in [0, A\varepsilon^{-1}]$ for any positive constant $A$.
- $\sin(\varepsilon x + \varepsilon) = \varepsilon x + \varepsilon + \mathcal{O}(\varepsilon^3)$, which is only uniform on a finite interval and, moreover, does not show any of the inherent periodicity. If we introduce $X = \varepsilon x$, we get the much better $\sin(X + \varepsilon) = \sin(X) + \mathcal{O}(\varepsilon)$ which is even uniform in $\mathbb{R}$.
- $e^{-x/\varepsilon} = 0 + o(\varepsilon^0)$ is a singular expansion on $x > 0$, but if we introduce $\xi = x/\varepsilon$ it becomes the regular expansion $e^{-\xi} = \mathcal{O}(1)$ on $\xi > 0$.
- On $x > 0$ we have $\frac{1}{2} \arctan(\frac{x}{\varepsilon}) + \sin(\varepsilon x)/(1 + x^2) = 1 + \varepsilon (x/(1 + x^2) - 2/\pi x) + \mathcal{O}(\varepsilon^2)$ in $x = 1 + \varepsilon^2 (\sin(X)/X^2 - 2/\pi X) + \mathcal{O}(\varepsilon^3)$. In $X = \varepsilon x$, and $= \frac{1}{2} \arctan(x) + \varepsilon^2 x + \mathcal{O}(\varepsilon^3)$ in $\xi = x/\varepsilon$.
- $e^{-e^3 \sin(x\sqrt{1 + \varepsilon})} = \sin x + \varepsilon x (\frac{1}{2} \cos x - \sin x) + \mathcal{O}(\varepsilon^2)$, which is only uniform on an interval $[0, A]$. This cannot be improved by a single other choice of independent variable. However, if we introduce three variables $x_1 = \varepsilon x$ and $x_2 = (1 + \varepsilon^2)x$, we get the much better $e^{-e^3 \sin x_2 - \frac{1}{4} e^3 x_2 e^{-e^3} \cos x_2 + \mathcal{O}(\varepsilon^4)}$.  

3. REGULAR PERTURBATION PROBLEMS

3 Regular perturbation problems

If a function \( \Phi(x; \varepsilon) \) is implicitly given by an equation (usually a differential equation with boundary conditions), say

\[
L(\Phi, x; \varepsilon) = 0 \quad \text{on a domain } D
\]

and both \( \Phi \) and \( L(\Phi, x; \varepsilon) \) have a regular asymptotic expansion on \( D \) with the same gauge functions, (3.1) is called a regular perturbation problem [15]. The shape functions \( \Phi_n \) are determined as follows.

We expand

\[
L(\Phi, x; \varepsilon) = \mu_0(\varepsilon) L_0(\Phi_0, x) + \mu_1(\varepsilon) L_1(\Phi_1, \Phi_0, x) + \mu_2(\varepsilon) L_2(\Phi_2, \Phi_1, \Phi_0, x) + \ldots = 0.
\]

(3.2)

According to theorem II.16 each term vanishes, and the sequence \( (\Phi_n) \) can be determined by induction:

\[
L_0(\Phi_0, x) = 0, \quad L_1(\Phi_1, \Phi_0, x) = 0, \quad L_2(\Phi_2, \Phi_1, \Phi_0, x) = 0, \quad \ldots
\]

(3.3)

It should be noted that in many interesting cases the problem is only regular after a suitable coordinate transformation. The major task when solving the problem is then to find this scaled or shifted coordinate. Practically important solution methods of this type are the method of slow variation, for geometrically stretched or slowly varying configurations, and the Lindstedt-Poincaré method, for solutions which are periodic in time with an unknown, \( \varepsilon \)-dependent period.

If (3.1) is not a regular perturbation problem, we call it a singular perturbation problem. Practically important solution methods for singular perturbation problems are the method of matched asymptotic expansions, where regular expansions exist locally but not in the whole region considered, and the method of multiple scales, where 2 or more distinct long and short length scales occur intertwined.

3.1 Method of slow variation

Suppose we have a function \( \phi(x; \varepsilon) \) of spatial co-ordinates \( x \) and a small parameter \( \varepsilon \), such that the typical variation in one direction, say \( x \), is of the order of length scale \( \varepsilon^{-1} \). We can express this behaviour most conveniently by writing \( \phi(x, y, z; \varepsilon) = \Phi(\varepsilon x, y, z; \varepsilon) \). Now if we were to expand \( \Phi \) for small \( \varepsilon \), we might, for example, get something like

\[
\Phi(\varepsilon x, y, z; \varepsilon) = \Phi(0, y, z; 0) + \varepsilon(x \Phi_x(0, y, z; 0) + \Phi_y(0, y, z; 0)) + \ldots
\]

which is only uniform in \( x \) on an interval \([0, L]\) if \( L = o(\varepsilon) \), and the inherent slow variation on the longer scale of \( x = \Theta(\varepsilon^{-1}) \) would be masked. It is clearly much better to introduce the scaled variable \( X = \varepsilon x \), and a (assumed) regular expansion of \( \Phi(X, y, z; \varepsilon) \)

\[
\Phi(X, y, z; \varepsilon) = \mu_0(\varepsilon) \phi_0(X, y, z) + \ldots
\]

(3.4)

now retains the slow variation in \( X \) in the shape functions of the expansion.

This situation frequently happens when the geometry involved is slender [34]. The theory of one-dimensional gas dynamics, lubrication flow, or sound propagation in horns (Webster’s equation) are important examples, although they are usually derived not systematically, without explicit reference to the slender geometry. We will illustrate the method by an example of heat flow in a varying bar.
Example II.23 Consider the stationary problem of the temperature distribution $T$ in a long heat-conducting bar with surface normal $n_S$ and slowly cross section $A$. The bar is kept at a temperature difference such that a given heat flux is maintained, but is otherwise isolated. As there is no leakage of heat, the flux is constant. With spatial coordinates made dimensionless on a typical bar radius, we have the following equations and boundary conditions

$$\nabla^2 T = 0, \quad \nabla T \cdot n_S = 0, \quad \int_A \frac{\partial T}{\partial x} d\sigma = Q. \quad (3.5)$$

Integrate $\nabla^2 T$ over a slice $x_1 \leq x \leq x_2$, and apply Gauss’ theorem, to find that the axial flux $Q$ is indeed independent of $x$. The typical length scale of diameter variation is assumed to be much larger than a diameter. We introduce the ratio between a typical diameter and this length scale as the small parameter $\varepsilon$, and write for the bar surface

$$S = r - R(X, \theta) = 0, \quad X = \varepsilon x. \quad (3.6)$$

By writing $R$ as a continuous function of slow variable $X$, rather than $x$, we have made our formal assumption of slow variation explicit in a convenient and simple way, since $\frac{\partial}{\partial x} R = \varepsilon R_X = \mathcal{O}(\varepsilon)$.

The crucial step will now be the assumption that the temperature is only affected by the geometric variation induced by $R$. Any initial or entrance effects are ignored or have disappeared. As a result the temperature field $T$ is a function of $X$, rather than $x$, and its axial gradient scales on $\varepsilon$, as $\frac{\partial}{\partial x} T = \mathcal{O}(\varepsilon)$.

Introduce the gradient $\nabla S$ and the transverse gradient $\nabla_S$

$$\nabla S = -\varepsilon R_X e_x + e_r - r^{-1} R_\theta e_\theta, \quad \nabla_S = e_r - r^{-1} R_\theta e_\theta. \quad (3.7)$$

At the bar surface $S = 0$ the gradient $\nabla S$ is a vector normal to the surface, while the transverse gradient $\nabla_S$, directed in the plane of a cross section $X = \text{const.}$, is normal to the circumference $S(X = c, r, \theta) = 0$.

Inside the bar we have the rescaled heat equation

$$\varepsilon^2 T_{XX} + \nabla^2_{\perp} T = 0. \quad (3.8)$$

At the wall the boundary condition of vanishing heat flux is

$$\nabla T \cdot \nabla S = \varepsilon^2 T_X S_X + \nabla_{\perp} T \cdot \nabla_S = 0 \quad \text{at} \quad S = 0. \quad (3.9)$$

The flux condition, for lucidity rewritten with $Q = \varepsilon q$, is given by

$$\int_A \frac{\partial T}{\partial x} d\sigma = q. \quad (3.10)$$

This problem is too difficult in general, so we try to utilize in a systematic manner the small parameter $\varepsilon$. Since the perturbation terms are $\mathcal{O}(\varepsilon^2)$, we assume the asymptotic expansion

$$T(X, r, \theta; \varepsilon) = T_0(X, r, \theta) + \varepsilon^2 T_1(X, r, \theta) + \mathcal{O}(\varepsilon^4). \quad (3.11)$$

After substitution in equation (3.8) and boundary condition (3.9), further expansion in powers of $\varepsilon^2$ and equating like powers of $\varepsilon$, we obtain to leading order a Laplace equation in $(r, \theta)$

$$\nabla_{\perp}^2 T_0 = 0 \quad \text{with} \quad \nabla_{\perp} T_0 \cdot \nabla_S = 0 \quad \text{at} \quad S = 0. \quad (3.12)$$

An obvious solution is $T_0 \equiv 0$. Since solutions of Laplace’s equation with vanishing normal derivatives at the boundary are unique up to a constant (here: a function of $X$), we have

$$T_0 = T_0(X). \quad (3.13)$$
3. REGULAR PERTURBATION PROBLEMS

We could substitute this directly in the flux condition, to find that \( AT_{0X} = q \). We do, however, not need this global conservation law, and it is interesting to see this result emerging from the equations. To obtain an equation for \( T_0 \) in \( X \) we continue with the \( \mathcal{O}(\varepsilon^2) \)-equation and corresponding boundary condition

\[
\nabla_\perp^2 T_1 + T_{0XX} = 0, \quad \nabla_\perp T_1 \cdot \nabla_\perp S = - T_{0X} S_X. \tag{3.14}
\]

The boundary condition can be rewritten as

\[
\nabla_\perp T_1 \cdot \mathbf{n}_\perp = \frac{T_{0X} R_X}{|\nabla_\perp S|} = \frac{T_{0X} R R_X}{\sqrt{R^2 + R_0^2}} \tag{3.15}
\]

where \( \mathbf{n}_\perp = \nabla_\perp S / |\nabla_\perp S| \) is the transverse unit normal vector. By integrating equation (3.14) over a cross section \( A \) of area \( A(X) \), using Gauss’ theorem, and noting that \( A = \int_0^{2\pi} \frac{1}{2} R^2 d\theta \), and that a circumferential line element is given by \( d\ell = (R^2 + R_0^2)^{1/2} d\theta \), we obtain

\[
\int_A \nabla_\perp^2 T_1 + T_{0XX} d\sigma = \int_{\partial A} \nabla_\perp T_1 \cdot \mathbf{n}_\perp d\ell + A \cdot T_{0XX}
= T_{0X} \int_0^{2\pi} R R_X d\theta + A \cdot T_{0XX} = A X T_{0X} + A \cdot T_{0XX} = \frac{d}{dX} \left( A \frac{d}{dX} T_0 \right) = 0. \tag{3.16}
\]

The finally obtained equation can be solved easily. Note that we recovered the conservation law of heat flux \( AT_{0X} = q \). Finally we have

\[
T_0(X) = \int_X^{x} \frac{q}{A(z)} dz + T_{\text{ref}} \tag{3.17}
\]

It should be noted we did not include in our analysis any boundary conditions at the ends of the bar. It is true that here the present method fails. The found solution is uniformly valid on \( \mathbb{R} \) (since \( R(X) \) is assumed continuous and independent of \( \varepsilon \)), but only as long as we stay away from any ends. Near the ends the boundary conditions induce transverse gradients of \( \mathcal{O}(1) \) which makes the prevailing length scale again \( x \), rather than \( X \). This region is asymptotically of boundary layer type, and should be treated differently (see below).

\( \square \)

Example II.24 Quasi 1-D gas dynamics. Consider a slowly varying duct with irrotational inviscid isentropic flow, described (in dimensionless form) by the velocity potential \( \phi \) and density \( \rho \) satisfying the equation for mass conservation and the compressible form of Bernoulli’s equation

\[
\nabla \cdot \left( \rho \nabla \phi \right) = 0, \quad \frac{1}{2} |\nabla \phi|^2 + \frac{\rho \gamma^{-1}}{2} = E. \tag{3.18}
\]

The parameter \( \gamma \) is a gas constant (1.4 for air) and \( E \) is a given problem parameter. Using the same notation as in the previous example, the duct wall is given by \( S(\varepsilon x, r, \theta) = 0 \), while at the impermeable wall \( \nabla \phi \cdot \nabla S = 0 \). The mass flux, the same at any cross section \( A \), is given by

\[
\int_A \rho \phi_1 d\sigma = F. \tag{3.19}
\]

Introduce the slow variable \( X = \varepsilon x \), and assume \( \phi \) and \( \rho \) to depend essentially on \( X \), rather than \( x \). Due to the scaling, the dimensionless axial flow velocity \( \phi_1 \), the density \( \rho \), the cross sectional area \( A \), the flux \( F \) and the thermodynamical constant \( E \) are \( \mathcal{O}(1) \). So we have to rescale \( \phi \) and write

\[
\phi(x, y, z; \varepsilon) = \varepsilon^{-1} \Phi(X, y, z; \varepsilon). \tag{3.20}
\]
The equations for $\Phi$ and $\rho$ become
\[ \varepsilon^2 \frac{\partial}{\partial x} \left( \rho \frac{\partial \Phi}{\partial x} \right) + \nabla \cdot \left( \rho \nabla \Phi \right) = 0, \quad \frac{1}{2} \Phi_x^2 + \frac{1}{2} \varepsilon^2 |\nabla \Phi|^2 + \frac{\rho \gamma - 1}{\gamma - 1} = E, \] (3.21a)
with boundary condition
\[ \nabla \Phi \cdot \nabla S = \varepsilon^2 \Phi_X S_X + \nabla \Phi \cdot \nabla S = 0 \quad \text{at} \quad S = 0. \] (3.21b)
We assume the expansion
\[ \Phi(X, y, z; \varepsilon) = \Phi_0(X, y, z) + O(\varepsilon^2), \quad \rho(X, y, z; \varepsilon) = \rho_0(X, y, z) + O(\varepsilon^2). \] (3.22)
From Bernoulli’s equation it follows that
\[ |\nabla \Phi_0|^2 = 0, \quad \Phi_0 = \Phi_0 \] (3.23)
Thus, we have a natural asymptotic series expansion for $f$ of the form
\[ f(t; \varepsilon) = \sum_{n=-\infty}^{\infty} A_n(\varepsilon) e^{i\omega_n(\varepsilon)t}. \] (3.24)
If amplitudes and frequency have an asymptotic expansion, say
\[ A_n(\varepsilon) = A_{n,0} + \varepsilon A_{n,1} + \ldots, \quad \omega(\varepsilon) = \omega_0 + \varepsilon \omega_1 + \ldots, \] (3.25)
we have a natural asymptotic series expansion for $f$ of the form
\[ f(t; \varepsilon) = \sum_{n=-\infty}^{\infty} A_{n,0} e^{i\omega_0 t} + \varepsilon \sum_{n=-\infty}^{\infty} \left( A_{n,1} + i n \omega_1 t A_{n,0} \right) e^{i\omega_0 t} + \ldots \] (3.26)
This expansion, however, is only uniform in $t$ on an interval $[0, T]$, where $T = o(\varepsilon^{-1})$. On a larger interval, for example $[0, \varepsilon^{-1}]$, the asymptotic hierarchy in the expansion becomes invalid, because $\varepsilon t = \Theta(1)$.

**Remark.**
The corresponding terms $\sim t \sin t, t \cos t$ are called “secular terms” (secular = occurring once in a century, saeculum = generation, referring to their astronomical origin).

It is therefore far better to apply first a coordinate transformation $\tau = \omega(\varepsilon)t$, introduce $F(\tau; \varepsilon) = f(t; \varepsilon)$, and expand $F$, rather than $f$, asymptotically.

In practical situations, the frequency $\omega$ is of course unknown, and to be found. When constructing the solution we have to allow for an unknown coordinate transformation
\[ \tau = (\omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \ldots) t \] (3.27)
(where we may take $\omega_0 = 1$) and determine the coefficients from the additional condition that the asymptotic hierarchy should be respected as long as possible. In other words, secular terms should not occur.
Example II.25 Consider the pendulum

\[ \ddot{\theta} + K^2 \sin(\theta) = 0, \quad \text{with } \theta(0) = \varepsilon, \quad \theta'(0) = 0. \quad (3.28) \]

After the transformation \( \tau = \cot \) and noting that \( \theta = O(\varepsilon) \), we have

\[ \omega^2 \theta'' + K^2 \left( \theta - \frac{1}{6} \theta^3 + \ldots \right) = 0, \quad (3.29) \]

We expand

\[ \omega = 1 + \varepsilon^2 \omega_1 + \ldots, \quad \theta = \varepsilon \theta_0 + \varepsilon^3 \theta_1 + \ldots \quad (3.30) \]

and find, after substitution, the equations for the first 2 orders

\[ \theta''_0 + K^2 \theta_0 = 0, \quad \theta_0(0) = 1, \quad \theta'_0(0) = 0 \quad (3.31a) \]
\[ \theta''_1 + K^2 \theta_1 = -2 \omega_1 \theta''_0 + \frac{1}{6} K \theta_0^3, \quad \theta_1(0) = 0, \quad \theta'_1(0) = 0 \quad (3.31b) \]

The solution for \( \theta_0 \) is obviously

\[ \theta_0 = \cos(K \tau), \quad (3.32) \]

leading to the following equation for \( \theta_1 \)

\[ \theta''_1 + K^2 \theta_1 = 2K \left( \omega_1 K + \frac{1}{16} \right) \cos(K \tau) + \frac{1}{16K} \cos(3K \tau). \quad (3.33) \]

Now it is essential to observe that the right-hand-side consists of two forcing terms: one with frequency \( 3K \) and one with \( K \), the resonance frequency of the left-hand-side. This resonance would lead to secular terms, as the solutions will behave like \( \tau \sin(K \tau) \) and \( \tau \cos(K \tau) \). Therefore, in order to suppress the occurrence of secular terms, the amplitude of the resonant forcing term should vanish, which yields the unknown \( \omega_1 \), and subsequently \( \theta_1 \)

\[ \omega_1 = -\frac{1}{16K} \quad \Rightarrow \quad \theta_1 = \frac{1}{48K} \left( \cos(K \tau) - \cos(K \tau)^3 \right) \quad (3.34) \]

\[ \square \]

4 Singular perturbation problems

4.1 Matched asymptotic expansions

Very often it happens that a simplifying limit applied to a more comprehensive model gives a correct approximation for the main part of the problem, but not everywhere: the limit is non-uniform. This non-uniformity may be in space, in time, or in any other variable. For the moment we think of non-uniformity in space, let’s say a small region near \( x = 0 \).

If this region of non-uniformity is crucial for the problem, for example because it contains a boundary condition, or a source, we are able to utilize the pursued limit and have to deal with the full problem (at least locally). This, however, does not mean that no use can be made of the inherent small parameter. The local nature of the non-uniformity itself gives often the possibility of another reduction. In such a case we call this a couple of limiting forms, “inner and outer problems”, and are evidence of the fact that we have apparently physically two connected but different problems as far as the dominating mechanism is concerned. Depending on the problem, we now have two simpler problems, serving as boundary conditions to each other via continuity or matching conditions.
CHAPTER II. PERTURBATION METHODS

Non-uniform asymptotic approximations

Suppose that a given sufficiently smooth function $\Phi(x; \epsilon)$, with $0 \leq x \leq 1$, $0 < \epsilon \leq \epsilon_1$, does not have a uniform limit $\epsilon \to 0, x \to 0$. Typically, such a function will depend, apart from $x$, on combinations like $x/\delta(\epsilon)$, where $\delta = o(1)$.

Assume that this function does not have a regular asymptotic expansion on the whole interval $[0, 1]$ but only on partial intervals $x \in [\eta(\epsilon), 1]$, where $\eta = o(1)$ and $\delta = o(\eta)$. We call this expansion the outer expansion, principally valid in the "$x = \mathcal{O}(1)$"-outer region [6, 17], but extendible to $[\mathcal{O}(\eta), 1]$.

$$\Phi(x; \epsilon) = \sum_{k=0}^{n} \mu_k(\epsilon)\phi_k(x) + o(\mu_n) \quad \epsilon \to 0, \quad x = \mathcal{O}(1).$$

(4.1)

Consider now the transformation to the stretched coordinate

$$\xi = \frac{x}{\delta(\epsilon)}. \quad (4.2)$$

Assume that the function $\Phi = \Psi(\xi; \epsilon)$ has a non-trivial regular asymptotic expansion on partial intervals $\xi \in [0, \zeta(\epsilon)/\delta(\epsilon)]$, where $\eta(\epsilon) < \zeta(\epsilon)$. We call this expansion the inner expansion, principally valid in the "$\xi = \mathcal{O}(1)$"-inner region, but extendible to $[0, \mathcal{O}(\zeta)]$.

$$\Phi(x; \epsilon) = \sum_{k=0}^{m} \lambda_k(\epsilon)\psi_k(\xi) + o(\lambda_m) \quad \epsilon \to 0, \quad \xi = \mathcal{O}(1).$$

(4.3)

Example II.26

$$\Phi(x; \epsilon) = \arctan(x/\epsilon) + \sin(x + \epsilon) = \frac{\pi}{2} + \sin x + \epsilon \cos x - \frac{\pi}{2} + \mathcal{O}(\epsilon^{3/2}) \quad \text{on} \quad \frac{1}{2}\epsilon^{1/2} \leq x \leq 1$$

$$\Psi(\xi; \epsilon) = \arctan(\xi) + \sin(\epsilon\xi + \epsilon) = \arctan(\xi) + \epsilon(\xi + 1) + \mathcal{O}(\epsilon^{3/2}) \quad \text{on} \quad 0 \leq \xi \leq 2\epsilon^{-1/2}$$

The adjective non-trivial is essential: the expansion must be “significant”, i.e. different from the outer-expansion in $\phi_n$ rewritten in the inner variable $\xi$. This determines the choice of the inner variable $\xi = x/\delta(\epsilon)$. The scaling $\delta(\epsilon)$ is the asymptotically largest gauge function with this property. We call the expansion for $\Psi$ the inner expansion or boundary layer expansion, the region $\xi = \mathcal{O}(1)$ or $x = \mathcal{O}(\delta)$ being the boundary layer with thickness $\delta$, and $\xi$ the boundary layer variable. Boundary layers may be nested, and may occur at internal points of the domain of $\Phi$. Then they are called “internal layers”. The assumption $\eta < \xi$, i.e. that inner and outer expansion may be extended to regions that overlap, is called the “overlap hypothesis”.

Suppose, $\Phi(x; \epsilon)$ has an outer-expansion

$$\Phi(x; \epsilon) = \sum_{k=0}^{n} \mu_k(\epsilon)\phi_k(x) + o(\mu_n)$$

(4.4)

and a boundary layer $x = \mathcal{O}(\delta)$ with inner-expansion

$$\Psi(\xi; \epsilon) = \sum_{k=0}^{m} \lambda_k(\epsilon)\psi_k(\xi) + o(\lambda_m)$$

(4.5)
and suppose that both expansions are complementary, i.e. there is no other boundary layer in between
\( x = \Theta(1) \) and \( x = \Theta(\delta) \), then the overlap-hypothesis says that both expansions represent the same
function in an intermediate region of overlap. This overlap region may be described by a stretched
variable \( x = \eta(\varepsilon)\sigma \), asymptotically in between \( \Theta(1) \) and \( \Theta(\delta) \), so: \( \delta \ll \eta \ll 1 \). In the overlap region
both expansions match, which means that asymptotically both expansions are equivalent and reduce
to the same expressions. A widely used and relatively simple procedure is Van Dyke’s matchings rule
[33, 6]: the outer-expansion, rewritten in the inner-variable, has a regular series expansion, which is equal
to the regular asymptotic expansion of the inner-expansion, rewritten in the outer-variable. Suppose that
\[
\sum_{k=0}^{n} \mu_k(\varepsilon) \varphi_k(\delta \xi) = \sum_{k=0}^{m} \lambda_k(\varepsilon) \eta_k(\xi) + o(\lambda_m) \quad (4.6a)
\]
\[
\sum_{k=0}^{n} \lambda_k(\varepsilon) \psi_k(x/\delta) = \sum_{k=0}^{n} \mu_k(\varepsilon) \theta_k(x) + o(\mu_n) \quad (4.6b)
\]
then the expansion of \( \eta_k \) back to \( x \)
\[
\sum_{k=0}^{n} \lambda_k(\varepsilon) \eta_k(x/\delta) = \sum_{k=0}^{n} \mu_k(\varepsilon) \zeta_k(x) + o(\mu_n) \quad (4.7)
\]
is such that \( \zeta_k = \theta_k \) for \( k = 0, \ldots, n \).

The idea of matching is very important because it allows one to move smoothly from one regime into
the other. The method of constructing local, but matching, expansions is therefore called “Matched
Asymptotic Expansions” (MAE) [17].

**Constructing asymptotic solutions**

The most important application of this concept of inner- and outer-expansions is that approximate solu-
tions of certain differential equations can be constructed for which the limit under a small parameter is
apparently non-uniform.

The main lines of argument for constructing a MAE solution to a differential equation + boundary
conditions are as follows. Suppose \( \Phi \) is given by the equation
\[
D(\Phi', \Phi, x; \varepsilon) = 0 \quad + \text{ boundary conditions,} \quad (4.8)
\]
where \( \Phi' = d\Phi/dx \). Then we try to construct an outer solution by looking for “non-trivial degenerations”
of \( D \) under \( \varepsilon \to 0 \), that is, find \( \mu_0(\varepsilon) \) and \( \nu_0(\varepsilon) \) such that
\[
\lim_{\varepsilon \to 0} v_0^{-1} D(\mu_0 \varphi_0', \mu_0 \varphi_0, x; \varepsilon) = D_0(\varphi_0', \varphi_0, x) = 0 \quad (4.9)
\]
has a non-trivial solution \( \varphi_0 \). A series \( \varphi = \mu_0 \varphi_0 + \mu_1 \varphi_1 + \cdots \) is constructed by repeating the process
for \( D - \nu_0 D_0 \), etc.

Suppose, the approximation is non-uniform. For example, not all boundary conditions can be satisfied.
Then we start looking for an inner-expansion if we have reasons to believe that the non-uniformity is of
boundary-layer type. Presence, location and size of the boundary layer(s) are now found by the “corre-
spondence principle”, that is the (heuristic) idea that if \( \Phi \) behaves somehow differently in the boundary
layer, the defining equation must also be essentially different. Therefore, we search for “significant de-
generations” or “distinguished limits” of \( D \). These are degenerations of \( D \) under \( \varepsilon \to 0 \), with scaled \( x \)
and \( \Phi \), that contain the most information, and without being contained in other, richer, degenerations.
Example II.27 Under the limit \( \varepsilon \to 0 \), the equation \( \varepsilon y' + y = \sin x \), \( y(0) = 1 \) reduces to \( y = \sin x \) with \( y(0) \neq 1 \). After the scaling \( x = \varepsilon \xi \), the equation reduces to the essentially different \( y_\varepsilon + y = 0 \).

The next step is then to select from these distinguished limits the one(s) allowing a solution that matches with the outer solution and satisfies any applicable boundary conditions.

Symbolically:

\[
\text{find } x_0, \delta(\varepsilon), \lambda(\varepsilon), \kappa(\varepsilon) \text{ with } x = x_0 + \delta \xi, \quad \Phi(x; \varepsilon) = \lambda(\varepsilon) \Psi(\xi; \varepsilon)
\]

such that \( B_0(\psi_0', \psi_0, \xi) = \lim_{\varepsilon \to 0} \kappa - 1 D(\delta^{-1} \lambda \Psi', \lambda \Psi, x_0 + \delta \xi; \varepsilon) \) has the “richest” structure, and there exists a solution of

\[
B_0(\psi_0', \psi_0, \xi) = 0 \quad (4.10)
\]
satisfying boundary and matching conditions. Again, an asymptotic expansion may be constructed inductively, by repeating the argument. It is of practical importance to note that the order estimate \( \lambda \) of \( \Phi \) in the boundary layer is often determined a posteriori by boundary or matching conditions.

Example II.28 A simple example to illustrate some of the main arguments is

\[
D(\phi', \phi, x; \varepsilon) = \varepsilon \frac{d^2 \phi}{dx^2} + \frac{d\phi}{dx} - 2x = 0, \quad \phi(0) = \phi(1) = 1. \quad (4.11)
\]

The leading order outer-equation is evidently (with \( \mu_0 = \nu_0 = 1 \))

\[
D_0 = \frac{d\phi_0}{dx} - 2x = 0 \quad (4.12)
\]

with solution

\[
\phi_0 = x^2 + A. \quad (4.13)
\]

The integration constant \( A \) can be determined by the boundary condition \( \phi_0(0) = 2 \) at \( x = 0 \) or \( \phi_0(1) = 2 \) at \( x = 1 \), but not both, so we expect a boundary layer at either end. By trial and error we find that no solution can be constructed if we assume a boundary layer at \( x = 1 \), so, inferring a boundary layer at \( x = 0 \), we have to use the boundary condition at \( x = 1 \) and find

\[
\phi_0 = x^2 + 1. \quad (4.14)
\]

The structure of the equation suggests a correction of \( O(\varepsilon) \), so we try the expansion

\[
\phi = \phi_0 + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \cdots. \quad (4.15)
\]

This results for \( \phi_1 \) into the equation

\[
\frac{d\phi_1}{dx} + \frac{d^2 \phi_0}{dx^2} = 0, \quad \text{with } \phi_1(1) = 0 \quad (\text{the } O(\varepsilon)\text{-term of the boundary condition}), \quad (4.16)
\]

which has the solution

\[
\phi_1 = 2 - 2x. \quad (4.17)
\]

Higher orders are straightforward:

\[
\frac{d\phi_n}{dx} = 0, \quad \text{with } \phi_n(1) = 0 \quad (4.18)
\]
leading to solutions $\psi_n \equiv 0$, and we find for the outer expansion
\[ \varphi = x^2 + 1 + 2\varepsilon(1 - x) + \mathcal{O}(\varepsilon^N) i \]  
(4.19)

We continue with the inner expansion, and find with $x_0 = 0$, $\varphi = \lambda \psi$, $x = \delta \xi$
\[ \varepsilon \lambda \frac{d^2 \psi}{\delta^2} + \lambda \frac{d \psi}{\delta} - 2\delta \xi = 0. \]  
(4.20)

Both from the matching ($\psi_{\text{outer}} \to 1$ for $x \downarrow 0$) and from the boundary condition ($\varphi(0) = 2$) we have to conclude that $\psi_{\text{inner}} = \mathcal{O}(1)$ and so $\lambda = 1$. Furthermore, the boundary layer has only a reason for existence if it comprises new effects, not described by the outer solution. From the correspondence principle we expect that new effects are only included if $(d^2 \psi/d\xi^2)$ is included. So $\varepsilon \delta^{-2}$ must be at least as large as $\delta^{-1}$, the largest of $\delta^{-1}$ and $\delta$. From the principle that we look for the equation with the richest structure, it must be exactly as large, implying a boundary layer thickness $\delta = \varepsilon$. Thus we have $\kappa = \varepsilon^{-1}$, and the inner equation
\[ \frac{d^2 \psi}{d\xi^2} + \frac{d \psi}{d\xi} - 2\varepsilon^2 \xi = 0. \]  
(4.21)

From this equation it would seem that we have a series expansion without the $\mathcal{O}(\varepsilon)$-term, since the equation for this order would be the same as for the leading order. However, from matching with the outer solution:
\[ \psi_{\text{outer}} \to 1 + 2\varepsilon + \varepsilon^2(\xi^2 - 2\xi) + \cdots \quad (x = \varepsilon \xi, \xi = \mathcal{O}(1)) \]  
(4.22)

we see that an additional $\mathcal{O}(\varepsilon)$-term is to be included. So we substitute the series expansion:
\[ \varphi = \psi_0 + \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \cdots. \]  
(4.23)

It is a simple matter to find
\[ \frac{d^2 \psi_0}{d\xi^2} + \frac{d \psi_0}{d\xi} = 0, \quad \psi_0(0) = 2 \implies \psi_0 = 2 + A_0(e^{-\xi} - 1) \]  
(4.24a)
\[ \frac{d^2 \psi_1}{d\xi^2} + \frac{d \psi_1}{d\xi} = 0, \quad \psi_1(0) = 0 \implies \psi_1 = A_1(e^{-\xi} - 1) \]  
(4.24b)
\[ \frac{d^2 \psi_2}{d\xi^2} + \frac{d \psi_2}{d\xi} = 2\xi, \quad \psi_2(0) = 0 \implies \psi_2 = \xi^2 - 2\xi + A_2(e^{-\xi} - 1) \]  
(4.24c)

where constants $A_0, A_1, A_2, \cdots$ are to be determined from the matching condition that outer expansion (4.19) for $x \to 0$:
\[ 1 + x^2 + 2\varepsilon - 2\varepsilon x + \cdots \]  
(4.25)

must be functionally equal to inner expansion (4.23) for $\xi \to \infty$:
\[ 2 - A_0 - \varepsilon A_1 + x^2 - 2\varepsilon x - \varepsilon^2 A_2 + \cdots. \]  
(4.26)

A full matching is obtained if we choose: $A_0 = 1, A_1 = -2, A_2 = 0$.

**Logarithmic switchback**

It is not always evident from just the structure of the equation what the necessary expansion will look like. Sometimes it is well concealed, and we are only made aware of an invalid initial choice by the warning of matching failure. In fact, it is also the matching process itself that reveals us the required sequence of scaling functions. An example of such a back reaction is known as **logarithmic switchback**.
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Example II.29 Consider the following problem for \( y = y(x; \varepsilon) \) on the unit interval.

\[
\varepsilon y'' + x(y' - y) = 0, \quad y(0; \varepsilon) = 0, \quad y(1; \varepsilon) = e. \tag{4.27}
\]

The outer solution appears to have the expansion

\[
y(x; \varepsilon) = y_0(x) + \varepsilon y_1(x) + \mathcal{O}(\varepsilon^2). \tag{4.28}
\]

By trial and error, the boundary layer appears to be located near \( x = 0 \), so the governing equations and boundary conditions are then

\[
y_0' - y_0 = 0, \quad y_0(1) = e, \tag{4.29a}
\]

\[
y_n' - y_n = -x^{-1} y_{n-1}' - y_n(1) = 0 \tag{4.29b}
\]

with general solution

\[
y_n(x) = A_n e^x + \int_\varepsilon^1 z^{-1} e^{x-z} y_{n-1}''(\varepsilon) \, dz \tag{4.30a}
\]

such that

\[
y_0(x) = e^x \tag{4.30b}
\]

\[
y_1(x) = -e^x \ln(x) \tag{4.30c}
\]

etc.

The boundary layer thickness is found from the scaling \( x = \varepsilon^m t \) (note that \( y = \mathcal{O}(1) \) because of the matching with the outer solution) leading to the significant degeneration of \( m = \frac{1}{2} \), or \( x = \varepsilon^{\frac{1}{2}} t \). The boundary layer equation for \( y(x; \varepsilon) = Y(t; \varepsilon) \) is thus

\[
Y'' + tY' - \varepsilon^{\frac{1}{2}} t Y = 0, \quad Y(0; \varepsilon) = 0. \tag{4.31}
\]

The obvious choice of expansion of \( Y \) in powers of \( \varepsilon^{\frac{1}{2}} \) is not correct, as the found solution does not match with the outer solution. Therefore, we consider the outer solution in more detail for small \( x \).

When \( x = \varepsilon^{\frac{1}{2}} t \), we have for the outer solution

\[
y(\varepsilon t; \varepsilon) = 1 + \varepsilon^{\frac{1}{2}} t - \frac{1}{2} \varepsilon \ln(\varepsilon) + \varepsilon \left( \frac{t^2}{2} - \ln(t) \right) - \frac{1}{4} \varepsilon^2 \ln(\varepsilon) - \varepsilon^2 t \ln(t) - \frac{1}{4} \varepsilon^2 \ln(\varepsilon)t^2 + \mathcal{O}(\varepsilon^2) \tag{4.32}
\]

So apparently we need at least

\[
Y(t; \varepsilon) = Y_0(t) + \varepsilon^{\frac{1}{2}} Y_1(t) + \varepsilon \ln(\varepsilon)Y_2(t) + \varepsilon Y_3(t) + o(\varepsilon) \tag{4.33}
\]

with equations and boundary conditions

\[
Y_0'' + tY_0' = 0, \quad Y_0(0) = 0, \tag{4.34a}
\]

\[
Y_1'' + tY_1' = tY_0, \quad Y_1(0) = 0, \tag{4.34b}
\]

\[
Y_2'' + tY_2' = 0, \quad Y_2(0) = 0, \tag{4.34c}
\]

\[
Y_3'' + tY_3' = tY_1, \quad Y_3(0) = 0, \tag{4.34d}
\]

with solution

\[
Y_0(t) = A_0 \text{erf} \left( \frac{t}{\sqrt{2}} \right) \tag{4.35a}
\]

\[
Y_1(t) = A_1 \text{erf} \left( \frac{t}{\sqrt{2}} \right) + Y_0'(t) + \int_0^t Y_0(z) \, dz = (A_1 + A_0t) \text{erf} \left( \frac{t}{\sqrt{2}} \right) + 2 \left( \frac{2}{\pi} \right)^{\frac{1}{2}} A_0 (e^{-t^2/2} - 1) \tag{4.35b}
\]

\[
Y_2(t) = A_2 \text{erf} \left( \frac{t}{\sqrt{2}} \right) \tag{4.35c}
\]

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Unfortunately, $Y_3$ cannot be expressed in closed form. However, for the demonstration it is sufficient to derive the behaviour of $Y_3$ for large $t$. As $\text{erf}(z) \to 1$ exponentially fast for $z \to \infty$, we obtain
\[ tY_1(t) = A_0t^2 + (A_1 - 2\left(\frac{2}{\pi}\right)^{\frac{1}{2}} A_0)t + \text{exponentially small terms} \quad (4.35d) \]
and thus after some algebra
\[ Y_3(t) = \frac{1}{2}A_0t^2 + (A_1 - 2\left(\frac{2}{\pi}\right)^{\frac{1}{2}} A_0)t - A_0\ln(t) + \ldots \quad (4.35e) \]
(The constant term cannot be determined because this depends on the value at $t = 0$.) For matching of the inner solution, we have to compare with expression (4.32), and have
\[ A_0 + \varepsilon^{\frac{1}{2}}(A_1 + A_0 - 2\left(\frac{2}{\pi}\right)^{\frac{1}{2}} A_0) + \varepsilon \ln(\varepsilon)A_2 + \varepsilon\left(\frac{1}{4} A_0\varepsilon^2 + (A_1 - 2\left(\frac{2}{\pi}\right)^{\frac{1}{2}} A_0)t - A_0\ln(t)\right) \equiv 1 + \varepsilon^{\frac{1}{2}}t - \frac{1}{2}\varepsilon \ln(\varepsilon) + \varepsilon\left(\frac{1}{2}t^2 - \ln(t)\right) \quad (4.36) \]
and indeed, a full matching is achieved with
\[ A_0 = 1, \quad A_1 = 2\left(\frac{2}{\pi}\right)^{\frac{1}{2}}, \quad A_2 = -\frac{1}{2}. \quad (4.37) \]

The role of matching

It is important to note that a matching is possible at all! Only a part of the terms can be matched by selection of the undetermined constants. Other terms are already equal, without free constants, and there is no way to repair a possibly incomplete matching here. This is an important consistency check on the found solution, at least as long as no real proof is available. If no matching appears to be possible, almost certainly one of the assumptions made with the construction of the solution have to be reconsidered. Particularly notorious are logarithmic singularities of the outer solution, as we saw above.

Summarizing: matching of inner- and outer expansion plays an important rôle in the following ways:
\begin{enumerate}
  \item it provides information about the sequence of order (gauge) functions \{\mu_k\} and \{\lambda_k\} of the expansions;
  \item it allows us to determine unknown constants of integration;
  \item it provides a check on the consistency of the solution, giving us confidence in the correctness.
\end{enumerate}

4.2 Multiple scales

Introduction

Suppose a function $\phi(x; \varepsilon)$ depends on more than one length scale acting together, for example $x$, $\varepsilon x$, and $\varepsilon^2 x$. Then the function does not have a regular expansion on the full domain of interest, $x \leq \Theta(\varepsilon^{-2})$ say.

It is not possible to bring these different length scales together by a simple coordinate transformation, like in the method of slow variation or the Lindsted-Poincaré method, or to split up our domain in subdomains like in the method of matched asymptotic expansions.

Therefore we have to find another way to construct asymptotic expansions, valid in the full domain of interest. The approach that is followed in the method of multiple scales ([22, 4]) is at first sight rather radical: the various length scales are temporarily considered as independent variables: $x_1 = x$, $x_2 = \varepsilon x$, $x_3 = \varepsilon^2 x$, and the original function $\phi$ is identified with a more general function $\psi(x_1, x_2, x_3; \varepsilon)$ depending on a higher dimensional independent variable.
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Example II.30

\[ \phi(x; \varepsilon) = A(\varepsilon) e^{-\varepsilon x} \cos(x + \theta(\varepsilon)) \]

becomes

\[ \psi(x_1, x_2; \varepsilon) = A(\varepsilon) e^{-x_2} \cos(x_1 + \theta(\varepsilon)). \]

Since this identification is not unique, we may add constraints such that this auxiliary function \( \psi \) does have a Poincaré expansion on the full domain of interest. After having constructed this expansion, it may be associated to the original function along the line \( x_1 = x, x_2 = \varepsilon x, x_3 = \varepsilon^2 x \).

The typical problems that are treated by the method of multiple scales are slowly varying waves, affected by small effects during a long time. Intuitively, it is clear that over a short distance (a few wave lengths) the wave only sees a constant conditions, and will propagate approximately as in the constant case, but over larger distances it will somehow have to change its shape in accordance with its new environment.

The technique, utilizing this difference between small scale and large scale behaviour is the method of multiple scales. As with most approximation methods, this method has grown out of practice, and works well for certain types of problems. Typically, the multiple scale method is applicable to problems with on the one hand a certain global quantity (energy, power) which is conserved or almost conserved and controls the amplitude, and on the other hand two rapidly interacting quantities (kinetic and potential energy) controlling the phase.

An illustrative example

We will illustrate the method by considering a damped harmonic oscillator

\[ \frac{d^2y}{dt^2} + 2\varepsilon \frac{dy}{dt} + y = 0 \quad (t \geq 0), \quad y(0) = 0, \quad \frac{dy(0)}{dt} = 1 \]  

(4.38)

with \( 0 < \varepsilon \ll 1 \). The exact solution is readily found to be

\[ y(t) = e^{-\varepsilon t} \sin(\sqrt{1 - \varepsilon^2} t)/\sqrt{1 - \varepsilon^2} \]

(4.39)

A naive approximation for small \( \varepsilon \) and fixed \( t \) would give

\[ y(t) = \sin t - \varepsilon t \sin t + \mathcal{O}(\varepsilon^2) \]

(4.40)

which appears to be not a good approximation for large \( t \) for the following reasons:

1) if \( t = \mathcal{O}(\varepsilon^{-1}) \) the second term is of equal importance as the first term and nothing is left over of the slow exponential decay;

2) if \( t = \mathcal{O}(\varepsilon^{-2}) \) the phase has an error of \( \mathcal{O}(1) \) giving an approximation of which even the sign may be in error.

In the following we shall demonstrate that this type of error occurs also if we construct a straightforward approximate solution directly from equation (4.38). However, knowing the character of the error, we may then try to avoid them. Suppose we can expand

\[ y(t; \varepsilon) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots. \]

(4.41)

Substitute in (4.38) and collect equal powers of \( \varepsilon \):

\[ \mathcal{O}(\varepsilon^0): \quad \frac{d^2y_0}{dt^2} + y_0 = 0 \quad \text{with} \quad y_0(0) = 0, \quad \frac{dy_0(0)}{dt} = 1, \]

\[ \mathcal{O}(\varepsilon^1): \quad \frac{d^2y_1}{dt^2} + y_1 = -2 \frac{dy_0}{dt} \quad \text{with} \quad y_1(0) = 0, \quad \frac{dy_1(0)}{dt} = 0, \]
then

\[ y_0(t) = \sin t, \quad y_1(t) = -t \sin t, \quad \text{etc.} \]

Indeed, the straightforward, Poincaré type, expansion (4.41) that is generated breaks down for large \( t \) when \( \varepsilon t \geq O(1) \). As is seen from the structure of the equations for \( y_n \), the quantity \( y_n \) is excited (by the “source”-terms \( -2d y_{n-1}/dt \)) in its eigenfrequency, resulting in resonance. The algebraically growing terms of the type \( t^n \sin t \) and \( t^n \cos t \) that are generated are called in this context: \textit{secular} terms.

Apart from being of limited validity, the expansion reveals nothing of the real structure of the solution: a slowly decaying amplitude and a frequency slightly different from 1. For certain classes of problems it is therefore advantageous to incorporate this structure explicitly in the approximation.

Introduce the slow time scale

\[ T = \varepsilon t \quad (4.42) \]

and identify the solution \( y \) with a suitably chosen other function \( Y \) that depends on both variables \( t \) and \( T \):

\[ y(t; \varepsilon) = Y(t, T; \varepsilon). \quad (4.43) \]

The underlying idea is the following. There are, of course, infinitely many functions \( Y(t, T; \varepsilon) \) that are equal to \( y(t, \varepsilon) \) along the line \( T = \varepsilon t \) in \((t, T)\)-space. So we have now some freedom to prescribe additional conditions. With the unwelcome appearance of secular terms in mind it is natural to think of conditions, chosen such that no secular terms occur when we construct an approximation.

Since the time derivatives of \( y \) turn into partial derivatives of \( Y \)

\[ \frac{dy}{dt} = \frac{\partial Y}{\partial t} + \varepsilon \frac{\partial Y}{\partial T}, \quad (4.44) \]

equation (4.38) becomes for \( Y \)

\[ \frac{d^2 Y}{dt^2} + Y + 2\varepsilon \left( \frac{\partial Y}{\partial t} + \frac{\partial^2 Y}{\partial t \partial T} \right) + \varepsilon^2 \left( \frac{\partial^2 Y}{\partial T^2} + 2 \frac{\partial Y}{\partial T} \right) = 0. \quad (4.45) \]

Assume the expansion

\[ Y(t, T; \varepsilon) = Y_0(t, T) + \varepsilon Y_1(t, T) + \varepsilon^2 Y_2(t, T) + \cdots \quad (4.46) \]

and substitute this into equation (4.45) to obtain to leading orders

\[ \frac{d^2 Y_0}{dt^2} + Y_0 = 0, \]
\[ \frac{d^2 Y_1}{dt^2} + Y_1 = -2 \frac{\partial Y_0}{\partial t} - 2 \frac{\partial^2 Y_0}{\partial t \partial T}. \]

\(^2\)From astronomical applications where these terms occurred for the first time in this type of perturbation series: \textit{secular} = occurring once in a century; \textit{saeculum} = generation.
with initial conditions
\[ Y_0(0, 0) = 0, \quad \frac{\partial}{\partial t} Y_0(0, 0) = 1, \]
\[ Y_1(0, 0) = 0, \quad \frac{\partial}{\partial t} Y_1(0, 0) = -\frac{\partial}{\partial T} Y_0(0, 0). \]
The solution for \( Y_0 \) is easily found to be
\[ Y_0(t, T) = A_0(T) \sin t \quad \text{with} \quad A_0(0) = 1, \] (4.47)
which gives a right-hand side for the \( Y_1 \)-equation of
\[ -2\left( A_0 + \frac{\partial A_0}{\partial T} \right) \cos t. \]
No secular terms occur (no resonance between \( Y_1 \) and \( Y_0 \)) if this term vanishes:
\[ A_0 + \frac{\partial A_0}{\partial T} = 0 \quad \rightarrow \quad A_0 = e^{-T}. \] (4.48)
Note (this is typical), that we determined \( Y_0 \) only on the level of \( Y_1 \), but without having to solve \( Y_1 \) itself.
The present approach is by and large the multiple scale technique in its simplest form. Variations on this theme are sometimes necessary. For example, we have not completely got rid of secular terms.

**Slowly varying fast time scale**

The method fails when the slow variation is due to external effects, like a slowly varying problem parameter.

**Example II.31** Consider the problem
\[ \ddot{x} + \kappa(\varepsilon t)^2 x = 0, \quad x(0; \varepsilon) = 1, \quad \dot{x}(0; \varepsilon) = 0. \] (4.49)
where \( \kappa = \Theta(1) \). It seems plausible to assume 2 time scales: a fast one \( \Theta(\kappa^{-1}) = \Theta(1) \) and a slow one \( \Theta(\varepsilon^{-1}) \). So we introduce next to \( t \) the slow scale \( T = \varepsilon t \), and rewrite \( x(t; \varepsilon) = X(t, T; \varepsilon) \). We expand \( X = X_0 + \varepsilon X_1 + \ldots \) and obtain \( X_0 = A_0(T) \cos(\kappa(T) t - \theta_0(T)) \). Suppressing secular terms in the equation for \( X_1 \) requires \( A_0' = \kappa' t - \theta_0' = 0 \), which is impossible.

In this case the fast variable is to be strained locally by a suitable strain function, as follows
\[ \tau = \int T \omega(z; \varepsilon) dz, \quad \text{where} \quad T = \varepsilon t, \] (4.50)
while for \( x(t; \varepsilon) = X(\tau, T; \varepsilon) \) we have
\[ \ddot{x} = \omega X_T + \varepsilon X_{TT} \quad \text{and} \quad \dddot{x} = \omega^2 X_{TT} + \varepsilon \omega X_{TT} + 2\varepsilon \omega X_{TT} + \varepsilon^2 X_{TT}. \] (4.51)
Example II.32. Reconsider the problem II.31. After expanding \( X = X_0 + \epsilon X_1 + \ldots \) and \( \omega = \omega_0 + \epsilon \omega_1 + \ldots \) we obtain

\[
\begin{align*}
\omega_0^2 X_{0tt} + \kappa^2 X_0 &= 0 \\
\omega_1^2 X_{1tt} + \kappa^2 X_1 &= -2\omega_0 \omega_1 X_{0tt} - \omega_1' X_{0t} - 2\omega_0 X_{0tT} 
\end{align*}
\] (4.52a, 4.52b)

The leading order solution is \( X_0 = A_0(T) \cos(\lambda(T) \tau - \theta_0(T)) \), where \( \lambda = \kappa/\omega_0 \). The right-hand side of (4.52b) is then

\[
2\omega_0 A_0 \lambda^2 (\omega_1 + \lambda' \tau - \theta_0') \cos(\lambda \tau - \theta_0) + (A_0 \lambda)^{-1} (\omega_0 A_0^2 \lambda^2)^{1/2} \sin(\lambda \tau - \theta_0).
\]

Suppression of secular terms requires \( \lambda' = 0 \). Without loss of generality we can take \( \lambda = 1 \), or \( \omega_0 = \kappa \). Then we need \( \omega_1 = \theta_0' \), which just yields that \( \lambda \tau - \theta_0 = \tau - \theta_0 = \epsilon^{-1} \int_0^T \omega_0(z) \, dz \). In other words, we can take \( \omega_1 = 0 \) and \( \theta_0 = a \) constant. Finally we have \( \omega_0 A_0^2 \lambda^2 = \kappa A_0^2 = a \) constant.

For linear wave-type problems we may anticipate the structure of the solution and assume the WKB hypothesis (see [4, 15])

\[
y(t; \epsilon) = A(T; \epsilon) e^{i\int_0^T \omega(t; \epsilon) \, dt}.
\] (4.53)

The method is again illustrated by the example of the damped oscillator (4.38). After substitution and suppressing the exponential factor, we get

\[
(1 - \omega^2)A + i \epsilon \left( 2\omega \frac{\partial A}{\partial T} + 2\omega_0 A + 2\omega A \right) + \epsilon^2 \left( \frac{\partial^2 A}{\partial T^2} + 2 \frac{\partial A}{\partial T} \right) = 0.
\]

Note that the secular terms are now not explicitly suppressed. The necessary additional condition is here that the solution of the present type exists (assumption 4.53), and that each higher order correction is no more secular than its predecessor. The solution is expanded as

\[
A(T; \epsilon) = A_0(T) + \epsilon A_1(T) + \epsilon^2 A_2(T) + \cdots \]
\[
\omega(T; \epsilon) = \omega_0(T) + \epsilon^2 \omega_2(T) + \cdots
\] (4.54)

Note that \( \omega_1 \) may be set to zero since the factor \( \exp(i \int_0^T \omega_1(\tau) \, d\tau) \) may be incorporated in \( A \). Substitute and collect equal powers of \( \epsilon \):

\[
\mathcal{O}(\epsilon^0): \quad (1 - \omega_0^2) A_0 = 0 \quad \Rightarrow \omega_0 = 1,
\]
\[
\mathcal{O}(\epsilon^1): \quad \frac{\partial A_0}{\partial T} + A_0 = 0 \quad \Rightarrow \quad A_0 = e^{-T},
\]
\[
\mathcal{O}(\epsilon^2): \quad 2i \left( \frac{\partial A_1}{\partial T} + A_1 \right) = (1 + 2\omega_2) e^{-T} \quad \Rightarrow \omega_2 = -\frac{1}{2}, \quad A_1 = 0.
\]

The solution that emerges is indeed consistent with the exact solution.

Example II.33. The air-damped resonator. In dimensionless form this is given by

\[
\frac{d^2 y}{dt^2} + \epsilon \frac{dy}{dt} \left| \frac{dy}{dt} \right| + y = 0, \quad \text{with} \quad y(0) = 1, \quad \frac{dy(0)}{dt} = 0.
\] (4.55)
CHAPTER II. PERTURBATION METHODS

By comparing the acceleration \( y'' \) with the damping \( \varepsilon y' \) it may be inferred that on a timescale \( \varepsilon t \) the influence of the damping is \( \mathcal{O}(1) \). So we conjecture a slow timescale \( \varepsilon t \), and split up the time dependence in two by introducing the slow timescale \( T \) and the dependent variable \( Y \)

\[
T = \varepsilon t, \quad y(t; \varepsilon) = Y(t, T; \varepsilon), \quad \frac{dy}{dt} = \frac{\partial Y}{\partial t} + \varepsilon \frac{\partial Y}{\partial T},
\]

and obtain for equation (4.55)

\[
\frac{\partial^2 Y}{\partial t^2} + Y + \varepsilon \left( 2 \frac{\partial^2 Y}{\partial t \partial T} + \frac{\partial Y}{\partial t} \frac{\partial |\partial Y|}{\partial t} \right) + \mathcal{O}(\varepsilon^2) = 0 \tag{4.56}
\]

\[
Y(0, 0; \varepsilon) = 0, \quad \left( \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T} \right) Y(0, 0; \varepsilon) = 0.
\]

The error of \( \mathcal{O}(\varepsilon^2) \) results from the approximation \( \frac{d}{dt} Y + \varepsilon \frac{d}{dt} Y \simeq \frac{d}{dt} Y \), and is of course only valid outside a small neighbourhood of the points where \( \frac{d}{dt} Y = 0 \). We expand

\[
Y(t, T; \varepsilon) = Y_0(t, T) + \varepsilon Y_1(t, T) + \mathcal{O}(\varepsilon^2)
\]

and find for the leading order

\[
\frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0, \quad \text{with} \quad Y_0(0, 0) = 1, \quad \frac{\partial}{\partial t} Y_0(0, 0) = 0 \tag{4.57}
\]

with solution

\[
Y_0 = A_0(T) \cos(t - \Theta_0(T)), \quad \text{where} \quad A_0(0) = 1, \quad \Theta_0(0) = 0.
\]

For the first order we have the equation

\[
\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = -2 \frac{\partial^2 Y_0}{\partial t \partial T} - \frac{\partial Y_0}{\partial t} \frac{\partial |\partial Y_0|}{\partial t} \]

\[
= 2 \frac{dA_0}{dT} \sin(t - \Theta_0) - A_0 \frac{d\Theta_0}{dT} \cos(t - \Theta_0)
\]

\[
+ A_0^2 \sin(t - \Theta_0) |\sin(t - \Theta_0)| \tag{4.58}
\]

with corresponding initial conditions. The secular terms are suppressed if the first harmonics of the right-hand side cancel. For this we use the Fourier series expansion

\[
\sin t |\sin t| = -\frac{8}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n + 1)t}{(2n - 1)(2n + 1)(2n + 3)} \tag{4.59}
\]

and we obtain the equations

\[
2 \frac{dA_0}{dT} + \frac{8}{3\pi} A_0^2 = 0 \quad \text{and} \quad \frac{d\Theta_0}{dT} = 0 \tag{4.60}
\]

with solution \( \Theta_0 = 0 \) and

\[
A_0(T) = \frac{1}{1 + \frac{4}{3\pi} T}. \tag{4.61}
\]

This approximation happens to be quite good. Comparison with a numerically obtained “exact” solution shows a relative error in the amplitude of less then \( 2 \cdot 10^{-3} \) for \( \varepsilon = 0.1 \).
Example II.34 Sound propagation in a slowly varying duct. Consider a hard-walled circular cylindrical duct with a slowly varying diameter described in polar coordinates \((x, r, \theta)\) as

\[ r = R(\varepsilon x) \quad (4.62) \]

with \(\varepsilon\) a dimensionless small parameter. In this duct we have an acoustic medium with constant mean pressure and sound speed \(c_0\). Sound waves of circular frequency \(\omega\) are described by Helmholtz’s equation

\[ \nabla^2 p + k^2 p = 0 \quad (4.63) \]

where \(p\) is the acoustical pressure perturbation, \(k = \omega/c_0\), and the boundary condition of a vanishing normal gradient at the wall yields

\[ \frac{\partial p}{\partial r} - \varepsilon R'(\varepsilon x) \frac{\partial p}{\partial x} = 0 \quad \text{at} \quad r = R(\varepsilon x). \quad (4.64) \]

For constant \(R\) and constant \(k\) the general solution can be built up from modes of the following type

\[ p = A J_m(\alpha_{m\mu} r) e^{-i m \theta} e^{-i k_{m\mu} x} \quad (4.65) \]

\[ \alpha_{m\mu} = j'_{m\mu}/R, \quad k_{m\mu}^2 = k^2 - \alpha_{m\mu}^2, \quad \text{Re}(k_{m\mu}) \geq 0, \quad \text{Im}(k_{m\mu}) \leq 0, \]

(where \(J_m\) denotes the \(m\)-th order Bessel function) and we assume for the present problem, following the previous section, that there are solutions close to these modes. We introduce the slow variable

\[ X = \varepsilon x \]

so that \(R = R(X)\), and we seek a solution of slowly varying modal type:

\[ p = A(X, r; \varepsilon) e^{-i m \theta} e^{-i k_{m\mu} x} \quad (4.66) \]

Since

\[ \frac{\partial^2 p}{\partial x^2} = (-\gamma^2 A - 2i \varepsilon \gamma \frac{\partial A}{\partial X} - i \varepsilon \gamma' A + \varepsilon^2 \frac{\partial^2 A}{\partial X^2}) \exp(\ldots) \]

we have for (4.63)

\[ \left[ -\gamma^2 A - 2i \varepsilon \gamma \frac{\partial A}{\partial X} - i \varepsilon \gamma' A + \varepsilon^2 \frac{\partial^2 A}{\partial X^2} + \frac{1}{r} \frac{\partial A}{\partial r} + \frac{m^2}{r^2} A + k^2 A \right] \exp(\ldots) = 0. \]

After suppressing the exponential factor, this is up to order \(O(\varepsilon)\)

\[ \mathcal{L}(A) = \frac{\varepsilon}{A} \frac{\partial A}{\partial X} (\gamma A^2), \quad \frac{\partial A}{\partial r} + \varepsilon R' \gamma A = 0 \quad \text{at} \quad r = R(X), \quad (4.67) \]

where we introduced for short the Bessel-type operator

\[ \mathcal{L}(A) = \frac{\partial^2 A}{\partial r^2} + \frac{1}{r} \frac{\partial A}{\partial r} + \left( k^2 - \gamma^2 - \frac{m^2}{r^2} \right) A \]

\[ \text{and rewrote the right-hand side in a form convenient later. Expand} \]

\[ A(X, r; \varepsilon) = A_0(X, r) + \varepsilon A_1(X, r) + \mathcal{O}(\varepsilon^2), \]

\[ \gamma(X; \varepsilon) = \gamma_0(X) + \mathcal{O}(\varepsilon^2), \]
substitute in (4.67), and collect like powers of \( \epsilon \), to obtain

\[
\mathcal{O}(1) : \quad \mathcal{L}(A_0) = 0, \quad \frac{\partial A_0}{\partial r} = 0 \quad \text{at} \quad r = R(X),
\]

(4.68)

\[
\mathcal{O}(\epsilon) : \quad \mathcal{L}(A_1) = \frac{i}{A_0} \frac{\partial}{\partial X} \gamma_0 A_0^2, \quad \frac{\partial A_1}{\partial r} = -i R' \gamma_0 A_0 \quad \text{at} \quad r = R(X).
\]

(4.69)

Since variable \( X \) plays no other rôle in (4.68) than that of a parameter, we have for \( A_0 \) the “almost-mode”

\[
A_0(X, r) = P_0(X) J_m(\alpha(X) r),
\]

(4.70)

\[
\alpha(X) = j' m \mu / R(X),
\]

\[
\gamma_\circ^2(X) = k^2 - \alpha^2(X), \quad \Re(\gamma) \geq 0, \quad \Im(\gamma) \leq 0,
\]

The amplitude \( P_0 \) is still undetermined, and follows from a solvability condition for \( A_1 \). As before, amplitude \( P_0 \) is determined at the level of \( A_1 \), without \( A_1 \) necessarily being known.

Multiply left- and right-hand side of (4.69) with \( r A_0 \) and integrate to \( r \) from 0 to \( R(X) \).

For the left-hand side we utilize the self-adjointness of \( \mathcal{L} \).

\[
\int_0^R r A_0 \mathcal{L}(A_1) \, dr = \int_0^R r A_0 \mathcal{L}(A_1) - r A_1 \mathcal{L}(A_0) \, dr = \left[ r A_0 \frac{\partial A_1}{\partial r} - r A_1 \frac{\partial A_0}{\partial r} \right]_0^R = -i \gamma_0 R' A_0^2.
\]

For the right-hand side we apply Leibnitz’s rule

\[
\int_0^R \frac{d}{dX} (i \gamma_0 A_0^2 r) \, dr = \frac{d}{dX} \int_0^R i \gamma_0 A_0^2 r \, dr - i \gamma_0 R' A_0^2.
\]

As a result

\[
\int_0^R r \gamma_0 A_0^2 \, dr = \left[ \frac{i \gamma_0 P_0^2}{\alpha} \left( r^2 - \frac{m^2}{\alpha^2} \right) J_m(\alpha r)^2 \right]_0^R = \frac{j' m \mu}{J_{m+1}} \left( 1 - \frac{m^2}{\alpha^2} \right) J_m(j' m \mu)^2 = \text{constant}
\]

or:

\[
P_0(X) = \frac{\text{constant}}{R(X) \sqrt{\gamma_\circ(X)}}
\]

(4.71)

**Example II.35 Ray acoustics in a temperature gradient.** When a sound wave propagates in free space through a medium that varies on a much larger scale than the typical wave length (typically: temperature gradients, or wind with shear), the same ideas of multiple scales may be applied. In contrast to the duct, where the wave is confined by the duct walls, the waves may now freely refract and follow curved paths. These paths are called rays.

Consider an infinite 3D medium with varying temperature (typical length scale \( L \)) but otherwise with a constant mean pressure, so that we have equation

\[
\nabla \cdot \left( \frac{1}{k^2} \nabla p \right) + p = 0
\]

(4.72)
4. SINGULAR PERTURBATION PROBLEMS

where \( k = k(\varepsilon x) = \omega / c_0(\varepsilon x) \), for a time harmonic sound field \( p \propto e^{i \omega t} \). The small parameter \( \varepsilon \) relates the typical wave length \( \lambda \sim 2\pi c_0 / \omega \) to \( L \), so \( \varepsilon \sim \lambda / L \). Assuming the field to be locally plane we try an approximate solution having the form of a plane wave but with slowly varying (real) amplitude \( A \) and phase \( \tau \)

\[
p(x) = A(X; \varepsilon) e^{-i \omega t / \varepsilon}
\]  

(4.73)

where \( X = \varepsilon x \) the slow variable. The surfaces

\[
\tau(X) = \varepsilon \omega t
\]

(4.74)

describe the propagating wave front. Note that the vector field \( \nabla \tau \) is normal to the surfaces \( \tau = \) constant. Define the operator

\[
\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)
\]

so that \( \nabla = \varepsilon \nabla \). Substitute (4.73) in (4.72):

\[
\nabla p := \left( \varepsilon \nabla A - i A \nabla \tau \right) e^{-i \omega t / \varepsilon},
\]

(4.75a)

\[
\nabla^2 p := \left( \varepsilon^2 \nabla^2 A - 2i \varepsilon \nabla A \cdot \nabla \tau - i \varepsilon A \nabla \tau^2 - A |\nabla \tau|^2 \right) e^{-i \omega t / \varepsilon},
\]

(4.75b)

to obtain

\[
(k^2 - |\nabla \tau|^2) A - i \varepsilon k^2 \nabla \cdot \left( \frac{\nabla^2 \tau}{k^2} \right) + \varepsilon^2 k^4 \nabla^4 \cdot \left( \frac{1}{k^2} \nabla A \right) = 0.
\]

(4.76)

Expand

\[
A(X; \varepsilon) = A_0(X) + \varepsilon A_1(X) + \mathcal{O}(\varepsilon^2)
\]

\[
\tau(X; \varepsilon) = \tau_0(X) + \mathcal{O}(\varepsilon^2)
\]

and collect like powers in (4.76). We find to leading order \( \tau_0 \) and \( A_0 \):

\[
|\nabla \tau_0|^2 = k^2
\]

(4.77)

\[
\nabla \cdot \left( \frac{A_0^2 \nabla \tau_0}{k^2} \right) = 0.
\]

(4.78)

Equation (4.77) is the eikonal equation, which determines the wave fronts and the ray paths. Equation (4.78) is called the transport equation and describes the conservation of wave action, which is here equivalent to conservation of energy [36]. It relates the amplitude variation to diverging or converging rays.

The eikonal equation is a nonlinear first order partial differential equation, of hyperbolic type, which can always be reduced to an ordinary differential equation along characteristics ([36, p.65]).
CHAPTER II. PERTURBATION METHODS

Exercises

1. Derive asymptotic solutions (for \( \varepsilon \to 0 \)) of the equation
   \[ \varepsilon x^3 - x + 2 = 0. \]

2. Derive step by step, by iteratively re-scaling \( x(\varepsilon) = \gamma_0(\varepsilon) X(\varepsilon) \)
   and balancing, re-scaling \( X(\varepsilon) = \gamma_1(\varepsilon) Y(\varepsilon) \)
   and balancing, etc., that a third order asymptotic solution (for \( \varepsilon \to 0 \)) of the equation
   \[ \log(\varepsilon x) + x = a \]
   is given by
   \[ x(\varepsilon) = \log \varepsilon^{-1} + \log(\log \varepsilon^{-1}) + a + o(1). \]

3. Derive Webster’s equation for sound of long wave length propagating in slowly varying horns,
   by the method of slender approximation. The reduced wave equation for pressure perturbations \( p \)
   and wavenumber \( k \) is given by
   \[ \nabla^2 p + k^2 p = 0, \]
   within a duct given in cylindrical polar coordinates by
   \[ S = r - R(\theta) = 0, \quad X = \varepsilon x, \quad \varepsilon \text{ is small.} \]
   The wave number is \( \mathcal{O}(\varepsilon) \), so we scale \( k = \varepsilon \kappa \). The duct wall is hard, so we have the boundary condition
   \[ \nabla p \cdot \nabla S = 0 \text{ at } S = 0. \]

4. Consider the \textit{van der Pol} equation for variable \( y = y(t; \varepsilon) \)
   in \( t \) and small parameter \( \varepsilon \):
   \[ y'' + y - \varepsilon (1 - y^2) y' = 0. \]
   Construct by using the Lindstedt-Poincaré method (“method of strained coordinates”) a first order approximation of a periodic solution.

5. Following example II.34, derive a multiple scales solution of sound waves in a slowly varying duct
   while also the sound speed is a slowly varying function of \( x \). The pertaining equation is therefore
   (4.72).

6. Consider a generalisation of example II.35.
   Acoustic perturbations in pressure \( p' \), density \( \rho' \), velocity \( v' \)
   and entropy \( s' \) of an inviscid mean flow, given by density \( \rho_0 \), velocity \( v_0 \), pressure \( p_0 \),
   sound speed \( c_0 \), entropy \( s_0 \) and constants \( C_p, C_V \) and \( \gamma \), are given by the linear equations
   \[ \frac{\partial \rho'}{\partial t} + v_0 \cdot \nabla \rho' + v' \cdot \nabla \rho_0 + \rho_0 \nabla \cdot v' + \rho' \nabla \cdot v_0 = 0 \]
   \[ \rho_0 \left( \frac{\partial v'}{\partial t} + v_0 \cdot \nabla v' + v' \cdot \nabla v_0 \right) + \rho' v_0 \cdot \nabla v_0 = -\nabla p' \]
   \[ \frac{\partial s'}{\partial t} + v_0 \cdot \nabla s' + v' \cdot \nabla s_0 = 0. \]
   \[ s' = \frac{C_V}{\rho_0} p' - \frac{C_p}{\rho_0} \rho', \quad \text{where} \quad \frac{C_p}{C_V} \frac{p_0}{\rho_0} = \gamma \frac{p_0}{\rho_0} = c_0^2. \]
4. SINGULAR PERTURBATION PROBLEMS

Assuming the perturbations to be harmonic in time, such that the typical wave length $c_0/\omega$ is short compared to the typical length scale $L$ of the mean flow variations, we introduce the ray approximation

$$p', \rho', v', s' = P(X), R(X), V(X), S(X) \times e^{i\omega t - i\tau(X, \varepsilon) / \varepsilon}$$

where $\omega \gg c_0/L$. Introduce a suitable small parameter $\varepsilon \sim c_0/\omega L$, and derive to leading order the equations for the ray amplitudes $P, R, V, S$ and the phase $\tau$. Eliminate $\nabla \tau$ to obtain an eikonal equation similar to (4.77).
### III Appendix

#### 1 Dimensionless numbers

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Archimedes</td>
<td>Ar</td>
<td>$g \Delta \rho L^3 / \rho v^2$</td>
<td>particles, drops or bubbles</td>
</tr>
<tr>
<td>Arrhenius</td>
<td>Arr</td>
<td>$E / RT$</td>
<td>chemical reactions</td>
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1. DIMENSIONLESS NUMBERS

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Bibliography


[5] dit moet nog


