PROBLEM SECTION

Solutions to the problems 875-879 (separate problems on separate sheets) should be sent to Mrs. J.G.A. Brands, Eindhoven University of Technology, Department of Mathematics and Computing Science, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, within six months after the appearance of this issue. The same rule applies to the March-July issue. Proposers of new problems are requested to enclose full solutions with their problems.

875. Let $a \in \mathbb{R}, a > 1$. Determine

$$f(x) := \lim_{N\to\infty} \frac{1}{\pi(a^N)} \sum_{n=1}^{N} \sum_{a^{n-1}<p\leq a^n} \frac{p}{a^n},$$

where $\pi(x)$ is the number of primes not exceeding x.

(J. VAN DE LUNE)

876. Let $a \in \mathbb{N}$. We define for $n \in \mathbb{N}$ the set

$$D_n := \{(x,y) \in \mathbb{R}^2 : |x| < \sqrt{n/a}, 0 < y \leq n - ax^2\}.$$

Let $A(n)$ denote the area of D_n, and $P(n)$ the number of (Gaussian) lattice points contained in D_n. Finally, define the 'error' $E(n)$ by $E(n) := A(n) - P(n)$.

1. Show that the (natural) density of $\{n \in \mathbb{N} | E(n) > 0\}$ equals $1 - 1/\sqrt{3}$.
2. Determine $\limsup_{n\to\infty} \frac{E(n)}{\sqrt{n}}$ and $\liminf_{n\to\infty} \frac{E(n)}{\sqrt{n}}$

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877. A well-known corollary of Murphy's Law is the fact that for a bicyclist, wind is more often a disadvantage than an advantage. In order to verify this empirical law theoretically, we investigate the following model:

An object moves in $\mathbb{R}^2$ with constant velocity $-\vec{v} = (-V, 0)$ where $V > 0$ in a uniform wind field with velocity $\vec{w} = (W\cos\theta, W\sin\theta)$, where $W \geq$

0, $0 \leq \theta \leq \pi$. The total effective velocity is $\vec{u} := \vec{v} + \vec{w}$. We assume the total air resistance $\vec{F}$ to be equal to $c|\vec{u}|\vec{u}$, where c is a positive constant. The drag adverse to the bicyclist's motion is the component of $\vec{F}$ in the x-direction, say $D(\theta, W, V)$. We introduce the quantity $\mu := W/V$.

1. Determine the angle $\theta =: \theta_1(\mu)$ for which

$$D(\theta, W, V) = D(\theta, 0, V).$$

2. Determine the value of μ for which $\theta_1(\mu)$ attains its maximum.

(S.W. RIENSTRA)

878. OLRY TERQUEM (1782-1862) called a subset of $\{1, 2, \ldots, n\}$ **alternating** if, when its elements are arranged in ascending order, the smallest element is odd, the next smallest element is even, the next is odd, etc. The empty set is counted as alternating. GEORGE ANDREWS (1938-) called a subset of $\{1, 2, \ldots, n\}$ **fat** if each of its elements is at least as large as the cardinality of the set. Again, the empty set is counted as fat. It is known that there are F_{n+2} alternating subsets of $\{1, 2, \ldots, n\}$, and also that there are F_{n+2} fat subsets of $\{1, 2, \ldots, n\}$, where F_k denotes the kth Fibonacci number (i.e., $F_1 := F_2 := 1$ and $F_k := F_{k-1} + F_{k-2}$ for $k \geq 3$.)

How many subsets of $\{1, 2, \ldots, n\}$ are both alternating and fat?

(C. ROUSSEAU)

879. Evaluate the sum

$$\sum_{k=1}^{2m+1} \left(1 - \sin \frac{2k\pi}{2m+1}\right)^{-1}$$

where m is a positive integer.

(SEUNG-JIN BANG)

2. Determine $\limsup_{n\to\infty}\frac{E(n)}{\sqrt{n}}$ and $\liminf_{n\to\infty}\frac{E(n)}{\sqrt{n}}$

(J. VAN DE LUNE)

Solutions by D. BRUIN, R.A. KORTRAM, J. V.D. LUNE, and P. SHIU.

Apart from J. V.D. LUNE the solvers are silent or somewhat unclear about the (well-known) asymptotic uniformity of the fractional part of $\sqrt{n/a}$.

SOLUTION by D. BRUIN.

$$A_n = \int_{-\sqrt{\frac{n}{a}}}^{\sqrt{\frac{n}{a}}} (n-ax^2)dx = [nx - \frac{1}{3}ax^3]_{-\sqrt{\frac{n}{a}}}^{\sqrt{\frac{n}{a}}} = \frac{4}{3}n\sqrt{\frac{n}{a}}.$$

$$P_n = \#\{(x,y)\in\mathbb{Z}^2 \mid |x| < \sqrt{\frac{n}{a}},\ 0 < y \leqq n-ax^2\} = \sum_{x=-\lambda_n}^{\lambda_n}(n-ax^2),$$

met $\lambda_n = \lceil \sqrt{\frac{n}{a}} - 1 \rceil$. Dus

$$P_n = n(2\lambda_n+1) - 2a\frac{2\lambda_n^3+3\lambda_n^2+\lambda_n}{6}.$$

Noem nu $S_n = \lceil \sqrt{\frac{n}{a}} - 1\rceil - \sqrt{\frac{n}{a}} = \lambda_n - \sqrt{\frac{n}{a}}$, dan geldt, $-1 \le \delta_n < 0$ en

$$\begin{aligned} P_n &= 2n(\sqrt{\frac{n}{a}}+\delta_n) + n - \frac{2}{3}a(\sqrt{\frac{n}{a}}+\delta_n)^3 - a(\sqrt{\frac{n}{a}}+\delta_n)^2 - \frac{1}{3}a(\sqrt{\frac{n}{a}}+\delta_n) \\ &= \frac{4}{3}n\sqrt{\frac{n}{a}} - a\sqrt{\frac{n}{a}}(\frac{1}{3}+2\delta_n+2\delta_n^2) - a(\frac{1}{3}\delta_n+\delta_n^2+\frac{2}{3}\delta_n^3). \end{aligned}$$

Dus, $E_n = A_n - P_n = a\sqrt{\frac{n}{a}}(\frac{1}{3}+2\delta_n+2\delta_n^2) + r_n$ met $|r_n| \leqq 2a$. Bekijk nu $x \mapsto \frac{1}{3}+2x+2x^2$ op $[-1,0)$; deze functie heeft maximum 1/3 voor $x=-1$ en minimum $-1/6$ voor $x=-\frac{1}{2}$. Het gebied waar de functie positief is heeft grootte $1-\frac{1}{3}\sqrt{3}$, dit geeft dus de 'kans' dat E_n positief is. Tenslotte geldt

$$\limsup_{n\to\infty}\frac{E_n}{\sqrt{n}} = \frac{1}{3}\sqrt{a} \quad \text{voor} \quad \delta_n : -1, \text{ m.a.w.} \quad n = k^2a \qquad \text{voor } k\in\mathbb{N},$$

$$\liminf_{n\to\infty}\frac{E_n}{\sqrt{n}} = -\frac{1}{6}\sqrt{a} \quad \text{voor} \quad \delta_n \sim -\tfrac{1}{2}, \text{ m.a.w.} \quad n = \lfloor (k+\tfrac{1}{2})^2 a\rfloor \quad \text{voor } k\in\mathbb{N}.$$

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0, $0 \leq \theta \leq \pi$. The total effective velocity is $\vec{u} := \vec{v} + \vec{w}$. We assume the total air resistance $\vec{F}$ to be equal to $c|\vec{u}|\vec{u}$, where c is a positive constant. The drag adverse to the bicyclist's motion is the component of $\vec{F}$ in the x-direction, say $D(\theta, W, V)$. We introduce the quantity $\mu := W/V$.

1. Determine the angle $\theta =: \theta_1(\mu)$ for which
$$D(\theta, W, V) = D(\theta, 0, V).$$
2. Determine the value of μ for which $\theta_1(\mu)$ attains its maximum.

(S.W. RIENSTRA)

Solutions by D. BRUIN, H.G. TER MORSCHE, S.W. RIENSTRA.

SOLUTION by H.G. TER MORSCHE.
By using the quantity $\mu = W/V$, the equation $D(\theta, W, V) = D(\theta, 0, V)$ can be written in the form

$$\sqrt{(1+\mu\cos\theta)^2 + (\mu\sin\theta)^2}(1+\mu\cos\theta) = 1$$

In order to solve the equation for θ as a function $\theta_1(\mu)$ of μ and to find the value μ for which $\theta_1(\mu)$ is maximal, we first substitute $x = 1 + \mu\cos\theta$, $y = \mu\sin\theta$ in this equation. Then, see the figure below, the solution $\theta_1(\mu)$ corresponds to the intersection point P of the circle $C : (x-1)^2 + y^2 = \mu^2$ and the curve $K : f(x,y) := x\sqrt{x^2+y^2} = 1$. Furthermore, the value μ for which $\theta_1(\mu)$ is maximal corresponds to the situation where the line MP is tangent to the curve K at the point P.

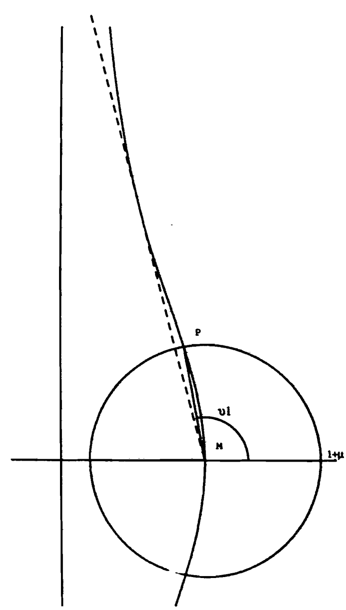

FIGURE 1.

It follows that the x-coordinate $x_1(\mu)$ of P must satisfy the equation

$$2x^3 + (\mu^2 - 1)x^2 - 1 = 0 \qquad (0 < x \leq 1)$$

One may solve this equation on several lucky ways or by using Cardano's formula.
The wanted solution $x_1(\mu)$ is given by

$$x_1(\mu) = \left(\sqrt[3]{1+\sqrt{1-\alpha}} + \sqrt[3]{1-\sqrt{1-\alpha}}\right)^{-1}$$

$$\alpha = (\frac{\mu^2 - 1}{3})^3$$

In case $\alpha > 1$ (i.e. $\mu > 2$) one may take the principal values of the roots in question.
Consequently, one has

$$\theta_1(\mu) = \arccos(\frac{x_1(\mu) - 1}{\mu})$$

Now, we return to the problem of finding the value for which $\theta_1(\mu)$ is maximal. This implies that grad f must be perpendicular to MP at the point $P = (x_1, y_1)$ which leads to the two equations

$$x_1\sqrt{x_1^2 + y_1^2} = 1$$

$$(2x_1^2 + y_1^2)(x_1 - 1) + x_1 y_1^2 = 0$$

By eliminating y_1 we get

$$x_1^4 - 2x_1 + 1 = (x_1 - 1)(x_1^3 + x_1^2 + x_1 - 1) = 0$$

The solution $x_1 \in (0, 1)$ is given by $x_1 = 0.543689....$.
Hence $\mu = \sqrt{(x_1 - 1)^2 + y_1^2} = \sqrt{1 - 2x_1 + 1/x_1^2} = 1.81538...$, and, finally

$$\theta_1 = 1.82448... \qquad (\approx 104.6°)$$

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