



VORTICAL PERTURBATIONS IN SHEAR FLOW, SCATTERED AT A HARD WALL – PRESSURE RELEASE WALL TRANSITION

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An analytically exact solution for the problem of low Mach number (*i.e.* incompressible) incident vorticity scattering at a hard - to - pressure release ($Z = 0$) wall transition is obtained using the Wiener-Hopf method. Harmonic vortical perturbations of inviscid linear shear flow are scattered at the wall transition, which results in a pressure-velocity field which is qualitatively different for low shear and high shear cases. The incompressible field produces an acoustic outer field, which can be determined for the low shear case, including a U_0^4 relation for the radiated power. This behaviour for $Z = 0$ is very similar to the asymptotic behaviour of the previously found solution for finite impedance [1]. This correspondence confirms the validity of this last solution.

1. Introduction

Inspired by the problem of acoustic radiation by turbulence along compliant surfaces [2], we considered in [1] the problem of scattering of harmonic ($e^{i\omega t}$) vortical perturbations in a mean flow of linear shear ($U = \sigma y$) along a wall, acoustically hard for $x < 0$ and acoustically treated by an impedance for $x > 0$. The approach was a formally exact Wiener-Hopf solution for the incompressible inner field, that was to be matched to an acoustic outer field. The main result, based on asymptotic analysis of the Fourier representation of the solution, showed a significantly different behaviour between the low shear ($\sigma < \omega$, relatively weak field) and the high shear ($\sigma > \omega$, relatively strong field) cases. Moreover, the non-decaying pressure field in the high shear case hampered the matching procedure, which urgently demanded additional confirmation of the found results. The present paper is a follow-up, in which we solved the same problem for the particular case of a pressure-release impedance ($Z = 0$). It appeared that, in addition to the Wiener-Hopf analysis, the Fourier integral representations of the incompressible field can now be expressed completely analytically (in terms of somewhat exotic functions like the incomplete Gamma function), while all the previous results could be confirmed and refined. In particular the significant difference between the low shear and high shear cases was recovered. Although much additional insight is obtained, the non-decaying high-shear pressure field remains acoustically an as yet not completely resolved difficulty. The field appears here driven by the mean flow and essentially coupled to the mean shear. Therefore, it is very different in character from seemingly similar 2D potential flow problems.

2. Model

The incident-field for the problem is obtained directly from [1]. For the record, we summarise the above introduction as follows. Consider the two-dimensional incompressible inviscid problem of

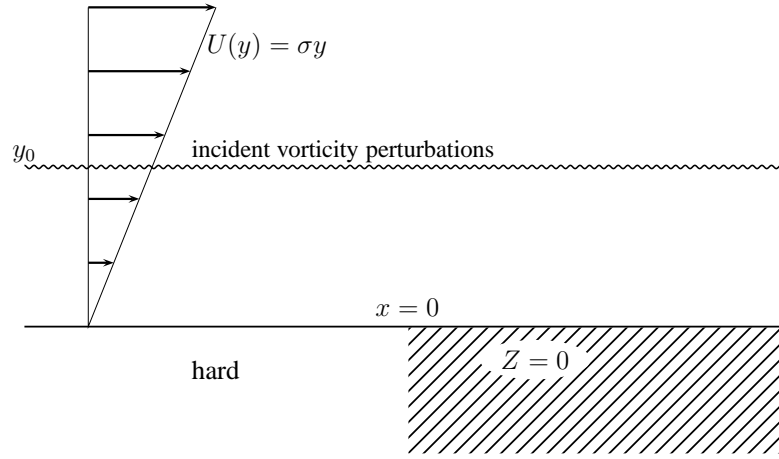


Figure 1: Sketch of the problem

perturbations of a linearly sheared mean flow with time dependent ($e^{i\omega t}$) vortex sheet along $y = y_0$ in $y > 0$ and a wall at $y = 0$ which is hard for $x < 0$ and pressure release ($Z = 0$) for $x > 0$ with $U(y) = \sigma y$; see Fig. 1. In this configuration we will have no contribution of a critical layer h_c or an instability like in [3]. We have a mass source placed at $x = x_0 \rightarrow -\infty$, $y = y_0$ which produce the downstream travelling vorticity that decays exponentially away from the line $y = y_0$ in the order $\sim e^{-k_0|y-y_0|-ik_0x}$. When the convected vorticity field hits the hard-to-soft wall transition point $x = 0$, it is scattered into a local pressure-velocity field that will radiate as sound into the far field.

3. Mathematical formulation

The governing equation of mass and momentum conservation written in frequency domain are

$$(1) \quad \begin{aligned} \rho_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0, \\ \rho_0 \left(i\omega + U \frac{\partial}{\partial x} \right) u + \rho_0 \frac{dU}{dy} v + \frac{\partial p}{\partial x} &= 0, \\ \rho_0 \left(i\omega + U \frac{\partial}{\partial x} \right) v + \frac{\partial p}{\partial y} &= 0. \end{aligned}$$

Boundary conditions at half planes $y = 0$ are vanishing velocity and pressure release wall ($Z = 0$) respectively, *i.e.*

$$(2) \quad v = 0 \quad \text{if } x < 0, \quad p = 0 \quad \text{if } x > 0$$

and an edge condition of vanishing energy flux from $(0, 0)$. The far field boundary conditions will be of vanishing velocity, but maybe not of vanishing pressure. The incident field (of the undulating vortex sheet at $y = y_0 = U_0/\sigma$) is given by [1, 4]

$$(3) \quad \begin{aligned} u_{\text{in}} &= U_0 e^{-ik_0x} \left[-\text{sign}(y - y_0) e^{-k_0|y-y_0|} + e^{-k_0(y+y_0)} \right], \\ v_{\text{in}} &= iU_0 e^{-ik_0x} \left[e^{-k_0|y-y_0|} - e^{-k_0(y+y_0)} \right], \\ p_{\text{in}} &= \frac{\sigma}{\omega} \rho_0 U_0^2 e^{-ik_0x} \left[(1 + k_0|y - y_0|) e^{-k_0|y-y_0|} - (1 + k_0(y - y_0)) e^{-k_0(y+y_0)} \right], \end{aligned}$$

with $k_0 = \omega/U_0$, $k_0 y_0 = \omega/\sigma$, and assumed to be scaled by a small amplitude. Fig. 2 shows velocities and pressure of a typical case.

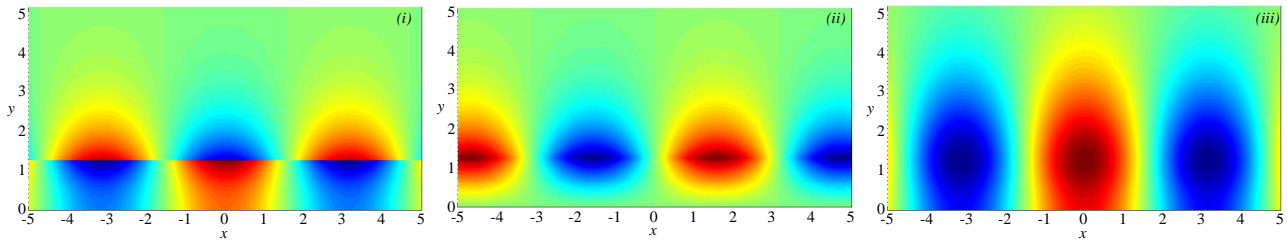


Figure 2: The initial field u_{in} , v_{in} and p_{in} . $\omega = 5$, $\sigma = 4$, $U_0 = 5$, $k_0 = 1$, $y_0 = 1.25$, (hard wall).

The triple $(u_{\text{in}}, v_{\text{in}}, p_{\text{in}})$ satisfies the differential equation, continuity of p_{in} and v_{in} across $y = y_0$, and the hard-wall boundary condition $v_{\text{in}} = 0$ at $y = 0$. The scattered field is thus due to the non-vanishing p_{in} along $y = 0$, $x > 0$. We split up the field in the incident part and the scattered part

$$(4) \quad u = u_{\text{in}} + \bar{u}, \quad v = v_{\text{in}} + \bar{v}, \quad p = p_{\text{in}} + \bar{p}.$$

After Fourier transformation in x (formally assuming the convergence of the integrals)

$$(5) \quad \bar{p}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{p}(y, k) e^{-ikx} dk,$$

(the same for \bar{u} and \bar{v}) we obtain the following set of equations

$$(6) \quad \rho_0(-ik\tilde{u} + \tilde{v}') = 0, \quad i\rho_0\Omega\tilde{u} + \rho_0\sigma\tilde{v} - ik\tilde{p} = 0, \quad i\rho_0\Omega\tilde{v} + \tilde{p}' = 0,$$

where $\Omega = \omega - kU$. The system of equations has two independent solutions, namely $\sim e^{\pm ky}$ [5, 6]. The one which is bounded for $y \rightarrow \infty$, is then

$$(7) \quad \tilde{u}(y) = kA(k) e^{-|k|y}, \quad \tilde{v}(y) = -i|k|A(k) e^{-|k|y}, \quad \tilde{p}(y) = \rho_0(\Omega - \text{sign}(\text{Re } k)\sigma)A(k) e^{-|k|y},$$

where $A(k)$ is to be determined, $|k| = \sqrt{k^2}$, and $\text{sign}(\text{Re } k) = k/|k|$ where $\sqrt{}$ denotes the principal value square root. $|k|$ has thus branch cuts along the imaginary axis given by $(-i\infty, 0)$ and $(0, i\infty)$.

4. Wiener-Hopf procedure

To facilitate the following Wiener-Hopf procedure, we introduce a small positive parameter ε and have an upper and a lower half plane, and a strip of overlap

$$(8) \quad \mathbb{C}^+ = \{k \in \mathbb{C} \mid \text{Im } k > -\varepsilon\}, \quad \mathbb{C}^- = \{k \in \mathbb{C} \mid \text{Im } k < \varepsilon\}, \quad S = \{k \in \mathbb{C} \mid -\varepsilon < \text{Im } k < \varepsilon\}.$$

The physical problem will be the limit $\varepsilon \rightarrow 0$ of a regularised problem with k_0 replaced by $k_0 - i\varepsilon \notin \mathbb{C}^+$ (an incident field $\sim e^{-ik_0x - \varepsilon x}$ slightly decaying with x) and $|k|$ replaced by the smoother function $|k| \simeq \sqrt{k^2 + \varepsilon^2}$, with branch cuts $(-i\infty, -i\varepsilon) \cup (i\varepsilon, i\infty)$ avoiding strip S (cf. [7]).

Introduce the half-range Fourier transforms

$$(9) \quad F_-(k) = \int_{-\infty}^0 \bar{p}(x, 0) e^{ikx} dx, \quad G_+(k) = \int_0^{\infty} \bar{v}(x, 0) e^{ikx} dx,$$

which are analytic in $\text{Im}(k) < 0$ and $\text{Im}(k) > 0$ respectively, and assumed to be analytic in \mathbb{C}^+ and \mathbb{C}^- . We have

$$(10) \quad G_+(k) = \int_0^{\infty} \bar{v}(x, 0) e^{ikx} dx = \int_{-\infty}^{\infty} \bar{v}(x, 0) e^{ikx} dx = -i|k|A(k).$$

$$\begin{aligned}
 (11) \quad F_-(k) &= \int_{-\infty}^0 \bar{p}(x, 0) e^{ikx} dx = \int_{-\infty}^{\infty} \bar{p}(x, 0) e^{ikx} dx + \int_0^{\infty} p_{\text{in}}(x, 0) e^{ikx} dx \\
 &= \rho_0 A(k) \frac{\omega|k| - \sigma k}{|k|} + 2i\rho_0 U_0^2 \frac{e^{-k_0 y_0}}{k - k_0} = \rho_0 A(k) |k| K(k) + 2i\rho_0 U_0^2 \frac{e^{-k_0 y_0}}{k - k_0},
 \end{aligned}$$

with Wiener-Hopf kernel

$$(12) \quad K(k) = \frac{\omega|k| - \sigma k}{|k|^2}.$$

From (10) and (11), we arrive at the Wiener-Hopf equation

$$(13) \quad F_-(k) = i\rho_0 G_+(k) K(k) + 2i\rho_0 U_0^2 \frac{e^{-k_0 y_0}}{k - k_0},$$

which is to be solved in the standard way [8] and writing

$$(14) \quad K(k) = \frac{K_+(k)}{K_-(k)},$$

where splitfunction K_+ is analytic and nonzero in \mathbb{C}^+ and K_- is analytic and nonzero in \mathbb{C}^- . The analysis for $\varepsilon \rightarrow 0$ requires some care. Since $K(k) \not\rightarrow 1$ at infinity, we cannot apply the usual recipe directly, and we have to modify the procedure in a way similar to Example 1.12 of Noble [8, p. 41-42]. Moreover, there is an essential difference between the high-shear case $\sigma > \omega$, where we have to remove a zero in S first. In the low-shear case $\sigma < \omega$, these splitfunctions are constructed in the usual way.

Finally we obtain the following

$$(15) \quad \begin{aligned} \sigma < \omega : \quad & K_+(k) = (\omega - \sigma)(k)_+^{-\frac{1}{2}-i\delta}, \quad K_-(k) = (k)_-^{\frac{1}{2}-i\delta}, \\ \sigma > \omega : \quad & K_+(k) = (\omega - \sigma)(k)_+^{-i\delta}, \quad K_-(k) = (k)_-^{1-i\delta}, \end{aligned} \quad \delta = \frac{1}{2\pi} \log \left| \frac{\omega + \sigma}{\omega - \sigma} \right|,$$

where $(k)_+^\alpha$ denotes the power function with $(1)_+^\alpha = 1$, the branch cut along the negative imaginary axis, and thus analytic in \mathbb{C}^+ , and $(k)_-^\alpha$ denotes the power function with $(1)_-^\alpha = 1$, the branch cut along the positive imaginary axis, and thus analytic in \mathbb{C}^- . Altogether, we can conclude that in S

$$(16) \quad K_-(k) F_-(k) - i\rho_0 K_+(k) G_+(k) = 2i\rho_0 U_0^2 e^{-k_0 y_0} \frac{K_-(k) - K_-(k_0)}{k - k_0} + 2i\rho_0 U_0^2 e^{-k_0 y_0} \frac{K_-(k_0)}{k - k_0},$$

where we isolated pole $k_0 \in \mathbb{C}^-$ from K_- . The parts that are analytic in \mathbb{C}^+ and in \mathbb{C}^- respectively, are via their equivalence in S each other's analytic continuations, and define an entire function E

$$\begin{aligned}
 (17) \quad E(k) &= K_-(k) F_-(k) - 2i\rho_0 U_0^2 e^{-k_0 y_0} \frac{K_-(k) - K_-(k_0)}{k - k_0} \\
 &= i\rho_0 K_+(k) G_+(k) + 2i\rho_0 U_0^2 e^{-k_0 y_0} \frac{K_-(k_0)}{k - k_0}.
 \end{aligned}$$

E can be determined from the condition of no energy flux at $r \rightarrow 0$ or $k \rightarrow \infty$, related to the edge condition for $(x, y) \rightarrow 0$. We conclude that $|G_+(k)| = O(k^{-\alpha})$ [1] for some $\alpha > 0$, so that for both low and high shear we have $E = 0$. The behaviour of the velocity to produce $E = 0$ is proved to be consistent with the solution. Hence we can write from (11) and (17)

$$\begin{aligned}
 (18) \quad F_-(k) &= 2i\rho_0 U_0^2 e^{-k_0 y_0} \frac{K_-(k) - K_-(k_0)}{(k - k_0) K_-(k)}, \\
 G_+(k) &= -2U_0^2 e^{-k_0 y_0} \frac{K_-(k_0)}{(k - k_0) K_+(k)}, \\
 A(k) &= -2iU_0^2 e^{-k_0 y_0} \frac{K_-(k_0)}{|k|(k - k_0) K_+(k)}.
 \end{aligned}$$

5. Analytic solution

$A(k)$ obtained from (18) can be substituted back into (7). This gives, with the inverse Fourier transform from (5) added to the initial field (3), and $\varepsilon = 0$, the formal solution of u , v and p

$$(19) \quad \begin{aligned} u &= u_{\text{in}} + U_0^2 K_-(k_0) e^{-k_0 y_0} \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\text{sign}(\text{Re } k)}{(k - k_0) K_+(k)} e^{-|k|y - ikx} dk, \\ v &= v_{\text{in}} - i U_0^2 K_-(k_0) e^{-k_0 y_0} \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1}{(k - k_0) K_+(k)} e^{-|k|y - ikx} dk, \\ p &= p_{\text{in}} + \rho_0 U_0^2 K_-(k_0) e^{-k_0 y_0} \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\Omega - \sigma \text{sign}(\text{Re } k)}{|k|(k - k_0) K_+(k)} e^{-|k|y - ikx} dk. \end{aligned}$$

The above integrals can be evaluated analytically exactly to produce the solution in terms of the regular and incomplete Gamma functions $\Gamma(\alpha)$ and $\Gamma(\alpha, z)$ [9]. As example, we show the result of \bar{u} for low shear. The expressions for high shear, and for \bar{p} and \bar{v} , are similar.

$$(20) \quad \begin{aligned} \bar{u} &= \frac{U_0}{\pi} e^{-k_0 y_0} \Gamma\left(\frac{3}{2} + i\delta\right) \left(\frac{\omega + \sigma}{\omega - \sigma}\right)^{\frac{1}{2}} \left[\frac{\omega}{\omega + \sigma} e^{-ik_0 z} \Gamma\left(-\frac{1}{2} - i\delta, -ik_0 z\right) \right. \\ &\quad \left. - \frac{\omega}{\omega - \sigma} e^{-ik_0 z^*} \Gamma\left(-\frac{1}{2} - i\delta, -ik_0 z^*\right) \right] - 2U_0 \frac{\omega}{\omega - \sigma} e^{-ik_0 x - k_0(y+y_0)} \quad \text{for } x > 0, \\ \bar{u} &= \frac{U_0}{\pi} e^{-k_0 y_0} \Gamma\left(\frac{3}{2} + i\delta\right) \left(\frac{\omega + \sigma}{\omega - \sigma}\right)^{\frac{1}{2}} \frac{\omega}{\omega + \sigma} \left[e^{-ik_0 z} \Gamma\left(-\frac{1}{2} - i\delta, -ik_0 z\right) \right. \\ &\quad \left. + e^{-ik_0 z^*} \Gamma\left(-\frac{1}{2} - i\delta, -ik_0 z^*\right) \right] \quad \text{for } x < 0, \end{aligned}$$

where $z = x + iy$ and $z^* = x - iy$, while the solution is singular if $\sigma = \omega$. Note that the solution indeed satisfies the boundary conditions (2) at $y = 0$, although we may have to invoke a small imaginary part for k_0 . If we take the large $r = |z|$ limit, we obtain the following expressions for low shear ($\sigma < \omega$)

$$(21) \quad \begin{aligned} \bar{u} &\sim i \frac{U_0}{\pi} e^{-k_0 y_0} e^{\frac{1}{4}\pi i} \Gamma\left(\frac{3}{2} + i\delta\right) \left(\frac{\omega + \sigma}{\omega - \sigma}\right)^{\frac{1}{4}} \left[\frac{\omega}{\omega + \sigma} (k_0 z)^{-\frac{3}{2} - i\delta} - \frac{\omega}{\omega - \sigma} (k_0 z^*)^{-\frac{3}{2} - i\delta} \right] \\ \bar{v} &\sim - \frac{U_0}{\pi} e^{-k_0 y_0} e^{\frac{1}{4}\pi i} \Gamma\left(\frac{3}{2} + i\delta\right) \left(\frac{\omega + \sigma}{\omega - \sigma}\right)^{\frac{1}{4}} \left[\frac{\omega}{\omega + \sigma} (k_0 z)^{-\frac{3}{2} - i\delta} + \frac{\omega}{\omega - \sigma} (k_0 z^*)^{-\frac{3}{2} - i\delta} \right] \\ \bar{p} &\sim \frac{\rho_0 U_0^2}{\pi} e^{-k_0 y_0} e^{\frac{1}{4}\pi i} \Gamma\left(\frac{1}{2} + i\delta\right) \left(\frac{\omega + \sigma}{\omega - \sigma}\right)^{\frac{1}{4}} \left[-(k_0 z)^{-\frac{1}{2} - i\delta} + (k_0 z^*)^{-\frac{1}{2} - i\delta} \right. \\ &\quad \left. - i\left(\frac{1}{2} + i\delta\right) \frac{\sigma}{\omega + \sigma} k_0 y (k_0 z)^{-\frac{3}{2} - i\delta} + i\left(\frac{1}{2} + i\delta\right) \frac{\sigma}{\omega - \sigma} k_0 y (k_0 z^*)^{-\frac{3}{2} - i\delta} \right] \end{aligned}$$

where the pressure decays with $O(r^{-\frac{1}{2}})$. For high shear ($\sigma > \omega$) we find

$$(22) \quad \begin{aligned} \bar{u} &\sim - \frac{U_0}{\pi} e^{-k_0 y_0} \Gamma(1 + i\delta) \left(\frac{\sigma + \omega}{\sigma - \omega}\right)^{\frac{1}{4}} \left[\frac{\omega}{\sigma + \omega} (k_0 z)^{-1 - i\delta} + \frac{\omega}{\sigma - \omega} (k_0 z^*)^{-1 - i\delta} \right] \\ \bar{v} &\sim - i \frac{U_0}{\pi} e^{-k_0 y_0} \Gamma(1 + i\delta) \left(\frac{\sigma + \omega}{\sigma - \omega}\right)^{\frac{1}{4}} \left[\frac{\omega}{\sigma + \omega} (k_0 z)^{-1 - i\delta} - \frac{\omega}{\sigma - \omega} (k_0 z^*)^{-1 - i\delta} \right] \\ \bar{p} &\sim i \frac{\rho_0 U_0^2}{\pi} e^{-k_0 y_0} \Gamma(i\delta) \left(\frac{\sigma + \omega}{\sigma - \omega}\right)^{\frac{1}{4}} \left[-(k_0 z)^{-i\delta} + (k_0 z^*)^{-i\delta} \right. \\ &\quad \left. + \delta \frac{\sigma}{\sigma + \omega} k_0 y (k_0 z)^{-1 - i\delta} + \delta \frac{\sigma}{\sigma - \omega} k_0 y (k_0 z^*)^{-1 - i\delta} \right]. \end{aligned}$$

Note that here the pressure does *not* decay with distance. This behaviour is essential for high shear and related to the edge condition in the origin. It is presumably an artefact of the model being 2D.

Shown in Fig. 3 and 4 are the solution fields $(\bar{u}, \bar{v}, \bar{p})$ for low and high shear cases respectively. The behaviour is qualitatively similar to that of a finite impedance Z case in [1]. The next step is to match the velocity solution in (21) and (22) to a compressible acoustic outer solution.

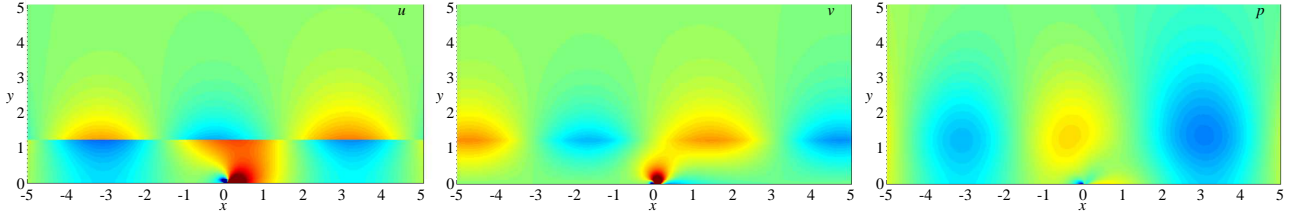


Figure 3: The solution fields u, v and p for low shear $\sigma = 4 < \omega = 5, y_0 = 1.25, k_0 = 1$ and $U_0 = 5$.

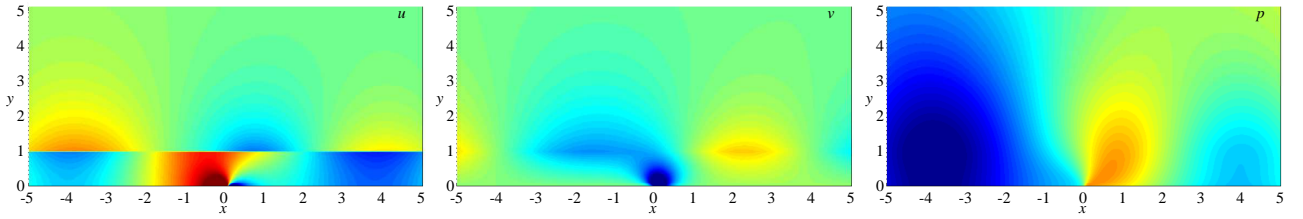


Figure 4: The solution fields u, v and p for high shear $\sigma = 5 > \omega = 4, y_0 = 1, k_0 = 0.8$ and $U_0 = 5$.

6. Acoustic outer field and asymptotic matching

By assuming the mean flow Mach number small (the initially linear shear profile has evidently to be curtailed by a smooth transition to a uniform profile), as well as the local Helmholtz number (the ratio between the acoustic wave number $\kappa = \omega/c_0$ and the hydrodynamic wave number $k_0 = \omega/U_0$) U_0/c_0 being small, the hydrodynamic inner problem is incompressible. This inner field, however, produces an acoustic outer field, which is compressible but with negligible mean flow. Then we have the Helmholtz (= reduced wave) equation for \bar{p} (or \bar{u} or \bar{v})

$$(23) \quad \nabla^2 \bar{p} + \kappa^2 \bar{p} = 0, \quad \kappa = \frac{\omega}{c_0}.$$

With a point source in $x = y = 0$, assuming a certain symmetry in r and θ (where $x = r \cos \theta$ and $y = r \sin \theta$), we search for solutions of the form

$$(24) \quad \bar{p}(r, \theta) = B_0 \gamma(r) \beta(\theta).$$

If we substitute this in the equations we find

$$(25) \quad \gamma'' + \frac{1}{r} \gamma' + \kappa^2 \gamma - \frac{\nu^2}{r^2} \gamma = 0, \quad \beta'' + \nu^2 \beta = 0,$$

such that $\beta(\theta) = B_1 e^{i\nu\theta} + B_2 e^{-i\nu\theta}$. Furthermore, due to the radiation condition,

$$(26) \quad \gamma(r) = m H_\nu^{(2)}(\kappa r) + n H_{-\nu}^{(2)}(\kappa r) = m H_\nu^{(2)}(\kappa r) + n e^{-\nu\pi i} H_\nu^{(2)}(\kappa r) = H_\nu^{(2)}(\kappa r),$$

with the relationship $H_{-\nu}^{(2)}(\kappa r) = e^{-i\nu\pi} H_\nu^{(2)}(\kappa r)$ [9]. Clearly, n can be taken zero and m equal to unity. The constants B_0, B_1, B_2 and ν are to be determined from the matching condition at $r \rightarrow 0$ where the Hankel function has the following asymptotic behaviour [9]

$$(27) \quad H_\nu^{(2)}(\kappa r) \simeq i\pi^{-1} \Gamma(\nu) \left(\frac{1}{2}\kappa r\right)^{-\nu} + i^{1+2\nu} \pi^{-1} \Gamma(-\nu) \left(\frac{1}{2}\kappa r\right)^\nu,$$

the second term of which can be ignored if $\text{Re}(\nu) > 0$, but is essential if ν is imaginary. We aim to match our outer solution (24) with the inner solutions (21) and (22). Since the incompressible field is necessarily derived for a linear mean flow $\propto y$, which evidently does not connect smoothly with a uniform mean flow, this matching cannot be accomplished in a strict asymptotic sense. However, like in [1], this can be repaired by adding a uniform flow on top of the shear flow, with an interface inbetween. As long as the backreaction from the interface is weak, the difference is small.

The approach we take is to use the fact that the scattered velocities \bar{u}, \bar{v} are harmonic functions which can be written as the gradient of a potential ϕ . For a bounded mean flow U , the pressure and radial velocity \bar{w} would then for large r be given by

$$(28) \quad \bar{p} = -\rho_0 \left(i\omega + U \frac{\partial}{\partial x} \right) \phi \simeq -i\rho_0 \omega \phi, \quad \bar{w} \simeq \frac{\partial}{\partial r} \phi.$$

The pressure obtained in this way is not exactly the same as given in (21) and (22) because of the presence of the mean flow σy -terms, but it is of the same form. Finally, the goal is to match ϕ with the acoustic solution. This, however, is only possible for the low shear case.

6.1 Low shear case matching

From (27) and (21), we have for $\sigma < \omega$ clearly $\nu = \frac{1}{2} + i\delta$, and

$$(29) \quad B_0 = \rho_0 U_0^2 e^{-k_0 y_0} e^{-\frac{1}{4}\pi i} \left(\frac{U_0}{2c_0} \right)^\nu \left(\frac{\omega + \sigma}{\omega - \sigma} \right)^{\frac{1}{4}}, \quad B_1 = \frac{\omega}{\omega - \sigma}, \quad B_2 = -\frac{\omega}{\omega + \sigma},$$

and so

$$(30) \quad \begin{aligned} \bar{p} &= B_0 H_\nu^{(2)}(\kappa r) (B_1 e^{i\nu\theta} + B_2 e^{-i\nu\theta}), \\ \bar{w} &= \frac{i}{\rho_0 c_0} B_0 H_\nu^{(2)'}(\kappa r) (B_1 e^{i\nu\theta} + B_2 e^{-i\nu\theta}). \end{aligned}$$

Using the far field behaviour

$$(31) \quad H_\nu^{(2)}(\kappa r) \sim \left(\frac{2}{\pi \kappa r} \right)^{\frac{1}{2}} e^{-i\kappa r + \frac{1}{2}i\nu\pi + \frac{1}{4}i\pi}, \quad H_\nu^{(2)'}(\kappa r) \sim -i \left(\frac{2}{\pi \kappa r} \right)^{\frac{1}{2}} e^{-i\kappa r + \frac{1}{2}i\nu\pi + \frac{1}{4}i\pi}$$

the time averaged radial acoustic intensity is then in the far field

$$(32) \quad \frac{1}{2} \text{Re}(\bar{p} \bar{w}^*) \simeq \frac{\rho_0 \kappa}{2\pi r} U_0^5 e^{-2k_0 y_0} \left(\frac{e^{2\delta\theta}}{(\omega + \sigma)^2} + \frac{e^{-2\delta\theta}}{(\omega - \sigma)^2} - \frac{2 \cos \theta}{\omega^2 - \sigma^2} \right).$$

Integrated over $0 < \theta < \pi$ we obtain the following expression for the radiated acoustic power

$$(33) \quad \int_0^\pi \frac{1}{2} \text{Re}(\bar{p} \bar{w}^*) r \, d\theta = \rho_0 c_0^3 y_0 \left(\frac{U_0}{c_0} \right)^4 \frac{e^{-2\omega/\sigma}}{\pi \delta} \left(\frac{\omega}{\sigma} - \frac{\sigma}{\omega} \right)^{-2}.$$

Apart from the expected $\rho_0 c_0^3 y_0$ and the assumed dimensionless (amplitude)² of the incident vorticity (3), the interesting part is $(U_0/c_0)^4$ times the function of ω/σ . (Note that $k_0 y_0 = \frac{\omega}{\sigma}$, $\delta = \delta(\frac{\omega}{\sigma})$.)

6.2 High shear case matching

A similar result for $\sigma > \omega$ is not possible. Since now $\nu = i\delta$ is imaginary, we need in \bar{p} (eq. 27) next to $r^{-i\delta}$ also a term $\propto r^{i\delta}$, which is essentially missing. For what it's worth, it is possible to match the velocities \bar{u} and \bar{v} , but the pressure remains difficult to interpret and is probably not meaningful.

With $\nu = 1 + i\delta$ and

$$(34) \quad B_0 = iU_0 e^{-k_0 y_0} \left(\frac{U_0}{2c_0} \right)^\nu \left(\frac{\sigma + \omega}{\sigma - \omega} \right)^{\frac{1}{4}}, \quad B_1 = \frac{\omega}{\sigma + \omega}, \quad B_2 = \frac{\omega}{\sigma - \omega}$$

we have then

$$\begin{aligned}
 \bar{u} &= B_0 H_\nu^{(2)}(\kappa r) (B_1 e^{-i\nu\theta} + B_2 e^{i\nu\theta}), \\
 \bar{v} &= i B_0 H_\nu^{(2)}(\kappa r) (B_1 e^{-i\nu\theta} - B_2 e^{i\nu\theta}), \\
 \bar{w} &= B_0 H_\nu^{(2)}(\kappa r) (B_1 e^{-i\nu\theta+i\theta} + B_2 e^{i\nu\theta-i\theta}), \\
 \bar{p} &= -i \rho_0 c_0 B_0 \int^{\kappa r} H_\nu^{(2)}(\xi) d\xi (B_1 e^{-i\nu\theta+i\theta} + B_2 e^{i\nu\theta-i\theta}).
 \end{aligned}
 \tag{35}$$

7. Conclusion

A systematic and analytically exact solution is obtained by means of the Wiener-Hopf technique of the problem of vorticity, convected by a linearly sheared mean flow and scattered by a hard wall - pressure-release wall transition. A qualitatively different behaviour of the hydrodynamic and far field sound is confirmed for low and high shear cases. A particular feature of the simplification $Z = 0$ is the fact that the Wiener-Hopf kernel can be split explicitly and the solution integrals can be evaluated analytically exactly, which allows much deeper insight into the problem. This enables us to find in rather detail the functional relationship of the hydrodynamic far field and hence the associated acoustic source strength.

If the mean shear is **low** ($\sigma < \omega$), the hydrodynamic far field decays like the inverse square root of the distance, and the interaction between edge and uniform flow is *weak*. The radiated acoustic power is found to vary with U_0^4 where U_0 is the mean flow velocity at the source position.

If the mean shear is **high** ($\sigma > \omega$), the hydrodynamic far field tends (in modulus) to a constant, and the interaction between edge and uniform flow is *strong*. A mean flow of uniform shear, ignoring any effects of the inevitably bounded velocity above, is apparently an inconsistent modelling assumption.

The results published for finite impedance Z in [1] are confirmed.

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