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Matrix geometric approach for random walks in the quadrant

Stella Kapodistria

AIQT Lecture 9

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The issue at hand

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The shorter queue problem: A numeric... using the matrix-geometric solution

Ilya Gertsbakh *

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Abstract

We consider a service system with two similar serv... arrival, joins the shorter queue. The state of the s... j⩾i, where i and j are the number of customers i... respectively. The stationary probability vector ... characteristics are obtained using the matrix-...

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SPECTRAL PROPERTIES OF THE TANDEM JACKSON NETWORK, SEEN AS A QUASI-BIRTH-AND-DEATH PROCESS

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Quasi-birth-and-death (QBD) processes with infinite “phase spaces” can exhibit unusual and interesting behavior. One of the simplest examples of such a process is the two-node tandem Jackson network, with the “phase” giving the state of the first queue and the “level” giving the state of the second queue.

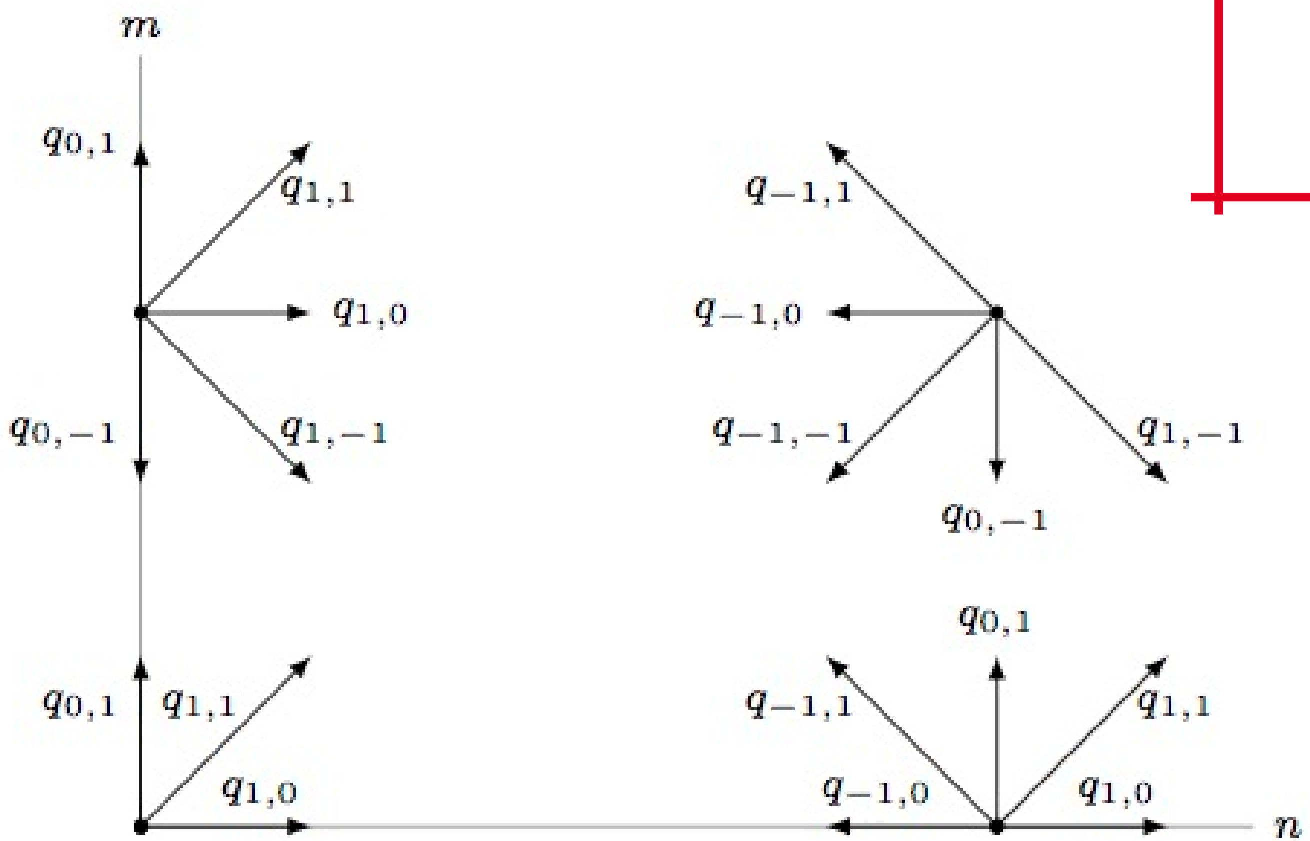
In this paper, we undertake an extensive analysis of the properties of this QBD. In particular, we investigate the spectral properties of Neuts’s R-matrix and show that the decay rate of the stationary distribution of the “level” process is not always equal to the convergence norm of R. In fact, we show that we can obtain any decay rate from a certain range by controlling only the transition structure at level zero, which is independent of R.

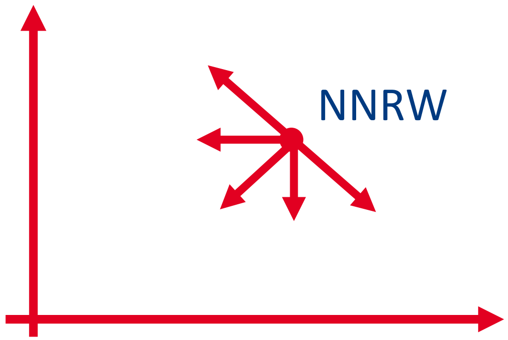
We also consider the sequence of tandem queues that is constructed by restricting the waiting room of the first queue to some finite capacity, and then allowing this capacity to increase to infinity. We show that the decay rates for the finite truncations converge to a value, which is not necessarily the decay rate in the infinite waiting room case.

Finally, we show that the probability that the process hits level n before level 0 given that it starts in level 1 decays at a rate which is not necessarily the same as the decay rate for the stationary distribution.

1. Introduction. A quasi-birth-and-death (QBD) process is a two-dimensional continuous-time Markov chain for which the generator has a block-tridiagonal structure. The first component of the QBD process is called the *level*, the second component the *phase*.

We consider the class of nearest neighbour random walks (NNRW)

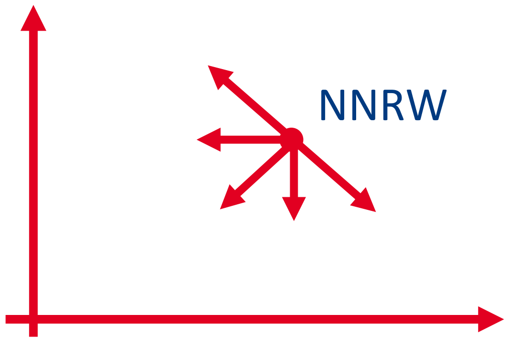




Main results

We consider the class of nearest neighbour random walks (NNRW) and we connect the

- Boundary value method approach
- Compensation approach
- Matrix geometric approach



Theorem 1

We extend the drift condition of Neuts for NNRW

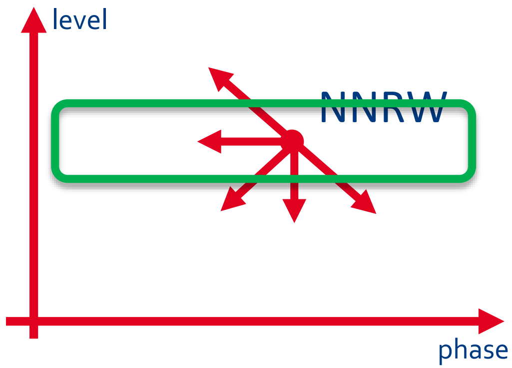
Theorem 2

We consider the class of NNRW and we calculate the eigenvalues and eigenvectors of $\boldsymbol{R}$ recursively, $A_1 + \boldsymbol{R}A_0 + \boldsymbol{R}^2 A_{-1} = 0$

Theorem 3

For the class of NNRW the infinite dimension rate matrix $\boldsymbol{R}$ is "diagonalizable" and we can numerically approximate $\boldsymbol{R}$ using spectral truncation.

Stability condition



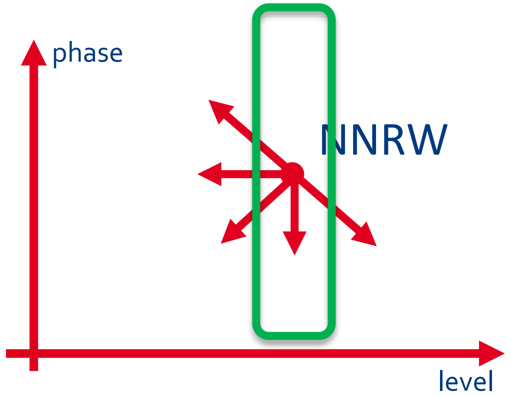
Let $A = A_{-1} + A_0 + A_1$ and x be the unique solution to

$$xA = 0$$

such that $x\vec{1} = 1$, with $\vec{1}$ a column vector of ones.

The stability condition is given by

$$xA_1\vec{1} < xA_{-1}\vec{1}$$



Let $\tilde{A} = \tilde{A}_{-1} + \tilde{A}_0 + \tilde{A}_1$ and x be the unique solution to

$$y\tilde{A} = 0$$

such that $y\vec{1} = 1$, with $\vec{1}$ a column vector of ones.

The stability condition is given by

$$y\tilde{A}_1\vec{1} < y\tilde{A}_{-1}\vec{1}$$

Stability condition

Proposition 3.1. *For the nearest neighbor random walk with generator matrices* $\mathbf{G}^V$ *or* $\mathbf{G}^H$, *defined in (8), the following holds:*

(i) The continuous-time Markov chain with generator $\boldsymbol{A}^V = \boldsymbol{A}^V_{-1} + \boldsymbol{A}^V_0 + \boldsymbol{A}^V_1$ *is ergodic if and only if* $M_x < 0$.

(ii) The continuous-time Markov chain with generator $\boldsymbol{A}^H = \boldsymbol{A}^H_{-1} + \boldsymbol{A}^H_0 + \boldsymbol{A}^H_1$ *is ergodic if and only if* $M_y < 0$.

(iii) The QBD drift condition

$$\boldsymbol{x}^H \boldsymbol{A}^V_1 \mathbb{1} < \boldsymbol{x}^V \boldsymbol{A}^V_{-1} \mathbb{1},$$

with $\boldsymbol{x}^V$ *the unique solution to* $\boldsymbol{x}^V \boldsymbol{A}^V = 0$ *such that* $\boldsymbol{x}^V \mathbb{1} = 1$, *where* $\mathbb{1}$ *a column vector of ones, is equivalent to* $M_x M'_y - M_y M'_x < 0$.

(iv) The QBD drift condition

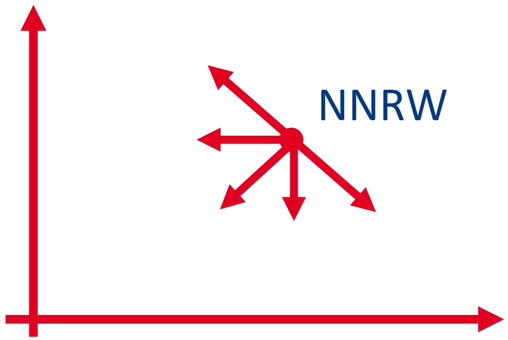
$$\boldsymbol{x}^H \boldsymbol{A}^H_1 \mathbb{1} < \boldsymbol{x}^H \boldsymbol{A}^H_{-1} \mathbb{1},$$

with $\boldsymbol{x}^H$ *the unique solution to* $\boldsymbol{x}^H \boldsymbol{A}^H = 0$ *such that* $\boldsymbol{x}^H \mathbb{1} = 1$, *where* $\mathbb{1}$ *a column vector of ones, is equivalent to* $M_y M''_x - M_x M''_y < 0$.

Nearest neighbour random walk

We consider the class of nearest neighbour random walks (NNRW):

- 1st quadrant
- Homogeneous nearest neighbour
- No transitions to N, NE and E



Then,

$$\pi(m,n) \sim c\alpha^m \beta^n \text{ as } m,n \to \infty$$

More concretely,

$$\pi(m,n) = \sum_i c_i \alpha_i^m \beta_i^n, m,n > 0$$

Boundary value method approach

First, introduce

$$\Pi(x,y)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\pi(m,n)x^m y^n$$

then

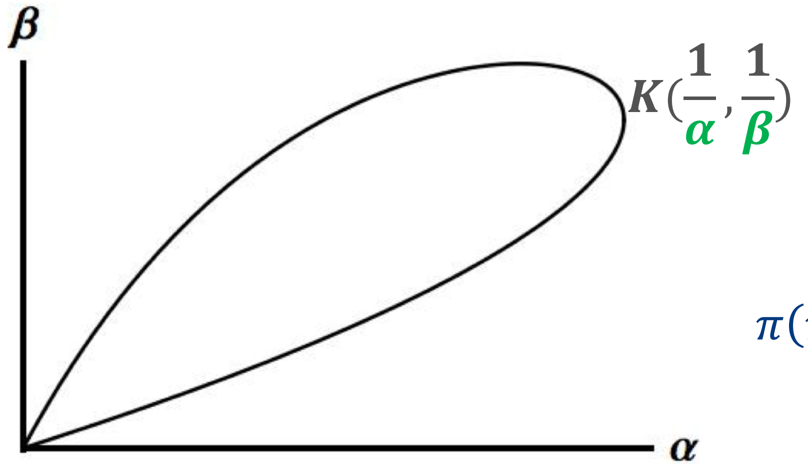
$$K(x,y)\Pi(x,y)=A(x,y)\Pi(x,0)+B(x,y)\Pi(0,y\,)+C(x,y)\Pi(0,0)$$

where $K(x,y), A(x,y), B(x,y), C(x,y)$ are known quadratic functions.

Choose $y=f(x)$, e.g. $y=\bar{x}$, and set $K(x,f(x))=0$

$$0=A(x,f(x))\Pi(x,0)+B(x,f(x))\Pi(0,f(x))+C(x,f(x))\Pi(0,0)$$

The above equation can be solved as a Riemann-Hilbert boundary value problem.



$$\pi(m,n)=\sum_i c_i \boldsymbol{\alpha}_i^m \boldsymbol{\beta}_i^n, m,n>0$$

Compensation approach

Aims at solving directly the balance equations of a random walk in the quadrant using a series (infinite or finite) of product-form solutions

Key idea:

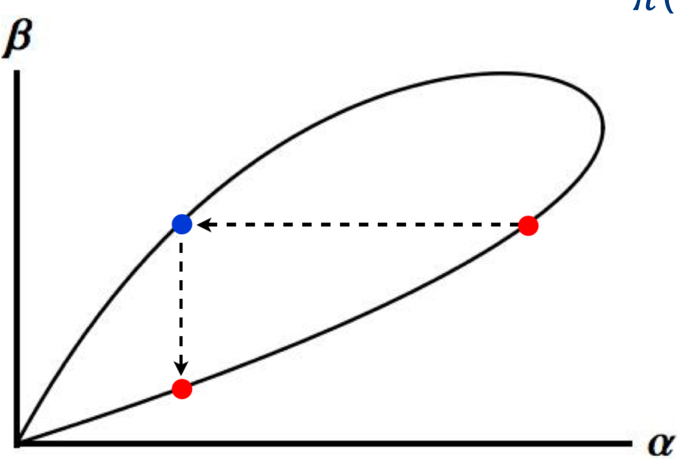
- Guess a product-form solution

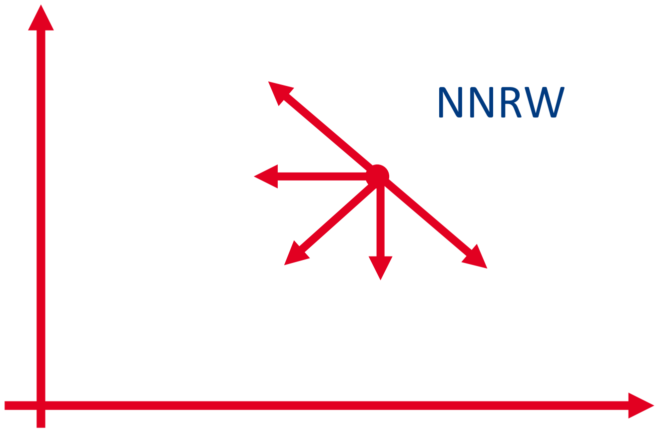
$$\alpha^m \beta^n$$

- Check if it satisfies the boundaries
- If not start compensating by adding new product-form terms

Solution

$$\pi(m,n) = \sum_i c_i \alpha_i^m \beta_i^n \,, m,n > 0$$





Matrix geometric approach

We know that

$$\boldsymbol{\pi}_m = \boldsymbol{\pi}_{m-1}\boldsymbol{R}$$

where $\boldsymbol{\pi}_m = (\pi(m,0) \;\; \pi(m,1) \;\; \dots)$ and $\pi(m,n) = \sum_i c_i \alpha_i^m \beta_i^n$, $m,n > 0$.

Then,

$$\Pi(x,y) = \boldsymbol{\pi}_0\boldsymbol{y} + \boldsymbol{\pi}_1(x^{-1}\boldsymbol{I} - \boldsymbol{R})^{-1}\boldsymbol{y}$$

where $\boldsymbol{y}' = (1 \; y \; y^2 \;\; \dots)$.

Substituting in the functional equation reveals

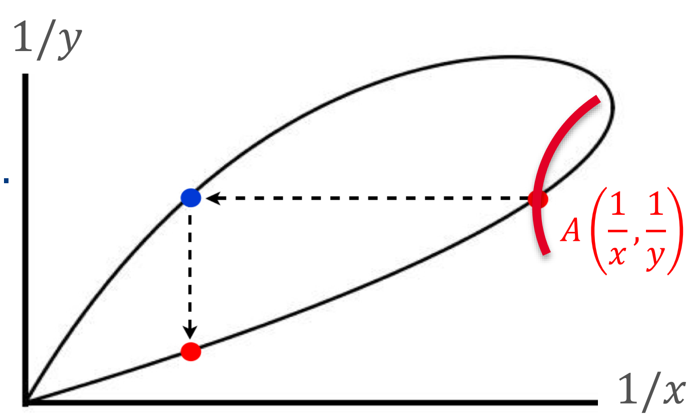
$$K(x,y)\Pi(x,y) = A(x,y)\Pi(x,0) + B(x,y)\Pi(0,y) + C(x,y)\Pi(0,0) \quad \Rightarrow$$

$$\boldsymbol{\pi}_1(x^{-1}\boldsymbol{I} - \boldsymbol{R})^{-1}[K(x,y)\boldsymbol{y} + A(x,y)\boldsymbol{e}]$$
$$= -\boldsymbol{\pi}_0\left[\big(K(x,y) + B(x,y)\big)\boldsymbol{y}' + \big(A(x,y) + C(x,y)\big)\boldsymbol{e}'\right]$$

So $x^{-1} = \alpha$ is an eigenvalue of matrix $\boldsymbol{R}$.

The terms *$y^{-1} = \beta$ are associated with the eigenvalues of $\boldsymbol{R}$.*

$$\alpha_0 \underbrace{\longrightarrow}_{\substack{\text{initial}\\ \text{solution}}} \beta_0 \underbrace{\longrightarrow}_{\substack{\text{vertical}\\ \text{compensation}}} \alpha_1 \underbrace{\longrightarrow}_{\substack{\text{horizontal}\\ \text{compensation}}} \beta_1 \quad \cdots$$



Matrix geometric approach

Theorem 1

The terms $\{\alpha_i\}$ constitute the different eigenvalues of the matrix $\boldsymbol{R}$, $A_1 + \boldsymbol{R}A_0 + \boldsymbol{R}^2 A_{-1} = 0$. For eigenvalue α_i the corresponding eigenvector of the matrix $\boldsymbol{R}$ is $\boldsymbol{h_i}$ with $h_{i,n} = c_i(\beta_{i-1}^n + f_i\beta_i^n)$.

Theorem 2

Spectral decomposition

$$\boldsymbol{R} = \boldsymbol{H}^{-1}\boldsymbol{D}\boldsymbol{H}$$

Truncated spectral decomposition

$$\boldsymbol{R_I} = \boldsymbol{H_I^{-1}}\boldsymbol{D_I}\boldsymbol{H_I}$$

Remark

The latter is equivalent to truncating

$$\pi(m,n) = \sum_{i=0}^{I} c_i \alpha_i^m \beta_i^n, m,n > 0$$

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TANDEM JACKSON
AND-DEATH PROCESS

AND P. G. TAYLOR[1]
nd Centre for Mathematics
ty of Melbourne

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Matrix geometric approach for random walks: Stability condition and equilibrium distribution

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ABSTRACT
In this paper, we analyze a sub-class of two-dimensional homogeneous nearest neighbor (simple) random walk restricted on the lattice using the matrix geometric approach. In particular, we first present an alternative approach for the calculation of the stability condition, extending the result of Neuts drift conditions[30] and connecting it with the result of Fayolle et al. which is based on Lyapunov functions.[13] Furthermore, we consider the sub-class of random walks with equilibrium distributions given as series of product forms and, for this class of random walks, we calculate the eigenvalues and the corresponding eigenvectors of the infinite matrix $\boldsymbol{R}$ appearing in the matrix geometric approach. This result is obtained by connecting and extending three existing approaches available for such an analysis: the matrix geometric approach, the compensation approach and the boundary value problem method. In this paper, we also present the spectral properties of the infinite matrix $\boldsymbol{R}$.

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Boundary value problem method; compensation approach; equilibrium distribution; matrix geometric approach; random walks; spectrum; stability condition

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1. Introduction

The objective of this work is to demonstrate how to obtain the stability condition and the equilibrium distribution of the state of a two-dimensional homogeneous nearest neighbor (simple) random walk restricted on the lattice using its underlying Quasi-Birth–Death (QBD) structure and the matrix geometric approach. This