AIQT : Lecture 4 notes

Generating function approach

The model

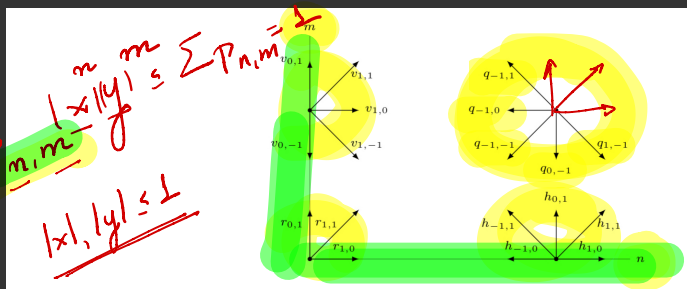


Figure 1: Transition rate diagram of the homogeneous simple random walk on the state space (n, m) with no transitions in the interior to the North, North-East, and East. Only the transitions at a few selected states are depicted as an indication.

The "result"

$$K(x,y)\,P(x,y) = A(x,y)\,P(x,0) + B(x,y)\,P(0,y) + C(x,y)\,P(0,0) \qquad (1)$$

with $P(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p_{n,m}\, x^n y^m$ $\quad |x|, |y| \le 1$

and $K(x,y)$, $A(x,y)$, $B(x,y)$, $C(x,y)$ known polynomials in x and y.

The "solution"

Step 1: Consider a zero, e.g.,

$(x,y) \in \{(x,y) \in \mathbb{C}\times\mathbb{C} : |x|, |y| \le 1 \text{ and } K(x,y) = 0\} = E$.

For $(x,y) \in E$, $P(x,y)$ is analytic and (1) reduces to

$$A(x,y)\,P(x,0) + B(x,y)\,P(0,y) + C(x,y)\,P(0,0) = 0 \qquad (2)$$

Step 2: Solve (2), e.g., express (2) as a boundary value problem.

Example 1: The 2×2 switch model

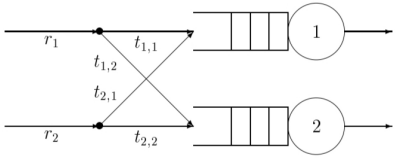


Figure 1: The 2 × 2 switch.

DTMC

Arrivals occur at the beg. of a slot

$t_{i,1} + t_{i,2} = 1 \quad i = 1, 2$

with rate transition diagram

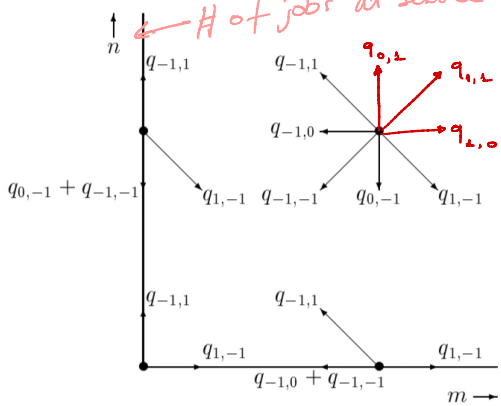


Figure 2: The one-step transition probabilities.

$$q_{1,-1} = r_1 r_2 t_{1,1} t_{2,1}, \qquad q_{0,0} = r_1 r_2 (t_{1,1} t_{2,2} + t_{1,2} t_{2,1}),$$
$$q_{0,-1} = r_1 (1 - r_2) t_{1,1} + r_2 (1 - r_1) t_{2,1}, \qquad q_{-1,1} = r_1 r_2 t_{1,2} t_{2,2},$$
$$q_{-1,0} = r_1 (1 - r_2) t_{1,2} + r_2 (1 - r_1) t_{2,2}, \qquad q_{-1,-1} = (1 - r_1)(1 - r_2).$$

Consider the simple, symmetric model: $r_1 = r_2 = p$ and $t_{ij} = \frac{1}{2}$

Let $p_{m,n}$ denote the joint stationary distribution of the queue lengths and

let $P(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p_{m,n} x^m y^n$ denote the corresponding PGF.

From the balance equations, it is straight forward to show that the PGF satisfies the following key functional equation

$$\big(xy - f(x,y)\big)\,P(x,y) = (y-1)\,f(x,0)\,P(x,0) + (x-1)\,f(0,y)\,P(0,y) + (x-1)(y-1)\,f(0,0)\,P(0,0) \tag{1}$$

with $f(x,y) = (1-p+\frac{p}{2}(x+y))^2$.

Step 1: Choose the right zero-tuples. We observe that Eq. (1) is symmetric in x and y. For this reason, we try $y=\bar{x}$ (conjugate) and select

$$\mathbb{E} = \{(x,y) \in \mathbb{C}\times\mathbb{C} : |x|,|y| \le 1,\ y=\bar{x} \text{ and } xy - f(x,y) = 0\}.$$

Note that $\mathbb{E}$ forms an ellipse.

Step 2: Eq. (1) now reduces to

$$0 = \frac{f(x,0)}{x-1}\,P(x,0) + \frac{f(0,\bar{x})}{\bar{x}-1}\,P(0,\bar{x}) + f(0,0)\,P(0,0), \quad x \neq 1 \tag{2}$$

Note that (i) $P(x,0) = P(0,x)$ because of the symmetry of the queueing system

(ii) $P(0,\bar{x}) = \sum_{n=0}^{\infty} \bar{x}^n P_{0,n} = \overline{P(0,x)}$.

(iii) $f(0,x) = f(x,0)$

(iv) $\lim_{x\to 1} f(x,0)\,P(x,0) = \left(1-\frac{p}{2}\right)^2 P(1,0)$ (*)

From (i) – (iii), Eq. (2) yields

$$\underbrace{\frac{f(x,0)}{x-1}\,P(x,0) + \frac{1}{2}\,f(0,0)\,P(0,0)}_{g(x)} = -\overline{\left(\frac{f(x,0)}{x-1}\,P(x,0) + \frac{1}{2}\,f(0,0)\,P(0,0)\right)}$$

$$\Rightarrow \quad g(x) = -\overline{g(x)} \quad \Rightarrow \quad 2\,\mathrm{Re}(g(x)) = 0, \quad x \in \mathbb{E}\setminus\{1\}$$

Question: Find a function $g(x)$ defined on the ellipse $E \setminus \{1\}$ such that (i) g is analytic inside the ellipse
(ii) g has a simple pole at 1
(iii) on the boundary of the ellipse $\mathrm{Re}(g(x)) = 0$ for $x \in E \setminus \{1\}$.

Answer: We know such a function, not on the ellipse but on the circle, say $\mathbb{D} = \{(x,y) \in \mathbb{C} \times \mathbb{C} : |x|, |y| \le 1,\ y = \bar{x} \text{ and } 1 - xy = 0\}$
The function

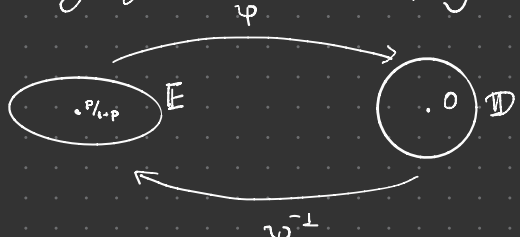
$$h(\omega) = c\,\frac{\omega+1}{\omega-1}$$

is analytic inside the disk, has a pole at 1 and on the circle, $|\omega|^2 = 1$,

$$\mathrm{Re}\left(h(\omega)\right) = \mathrm{Re}\left(c\,\frac{\omega+1}{\omega-1}\right) = c\,\mathrm{Re}\left(\frac{(\omega+1)\,\overline{(\omega-1)}}{|\omega-1|^2}\right) =$$

$$= \frac{c}{|\omega-1|^2}\,\mathrm{Re}\left(\underbrace{|\omega|^2}_{=1} - 1 - \omega + \bar{\omega}\right) = \frac{c}{|\omega-1|^2}\,\mathrm{Re}\left(-2\,\mathrm{Im}(\omega)\,i\right) = 0.$$

This is the so-called Dirichlet boundary value problem.
Note that in order to answer to the question, we need to map the circle on the ellipse E maintening its properties (preserving angles – conformal maping)



Then, $g(x) = h(\varphi(x)) = c\,\dfrac{\varphi(x)+1}{\varphi(x)-1}$. Lastly from (ii) and $\lim_{x\to 1}(x-1)\,g(x) = \dfrac{2c}{\varphi'(1)}$, we determine $\dfrac{2c}{\varphi'(1)} = f(1,0)\,P(1,0)$ ⊛.

Solution : We showed that

$$\frac{f(x,0)}{x-1} P(x,0) + \frac{1}{2} f(0,0) P(0,0) = c \frac{\varphi(x)+1}{\varphi(x)-1}$$

Then

$$P(x,y) = \frac{1}{K(x,y)} \left[\frac{f(x,0)}{x-1} P(x,0) + \frac{1}{2} f(0,0) P(0,0) + \frac{f(0,y)}{y-1} P(0,y) + \frac{1}{2} f(0,0) P(0,0) \right] (x-1)(y-1)$$

$$= \frac{1}{K(x,y)} \left[c \frac{\varphi(x)+1}{\varphi(x)-1} + c \frac{\varphi(y)+1}{\varphi(y)-1} \right] (x-1)(y-1) \quad (*)$$

$$= \frac{1}{K(x,y)} \xrightarrow{\varphi'(1)(1-p)} 2c \frac{\varphi(x)\varphi(y)-1}{(\varphi(x)-1)(\varphi(y)-1)} (x-1)(y-1)$$

(*) For $x=1$, Eq. (1) yields

$$P(1,y) = f(1,0) P(1,0) \frac{y-1}{y - f(1,y)}$$

For $y=1$, the above (together with the normalizing equation) yields

$$1 = f(1,0) P(1,0) \lim_{y \to 1} \frac{y-1}{y - f(1,y)} = f(1,0) P(1,0) \frac{1}{1-p}$$

$$\Rightarrow f(1,0) P(1,0) = 1-p.$$

Example 2: Simple combinatorial walk

Let $\{X_n, n=0,1,\dots\}$ denote the position of the walk evolving as follows

$$X_0 = 0$$

$$X_{n+1} = \begin{cases} 1, & \text{if } X_n = 0 \\ i \pm 1, & \text{if } X_n = i \end{cases}$$

Question: How many paths end at $X_n = i$ starting at $X_0 = 0$?

Answer: Let $p(n,i) = \#$ of paths ending at $X_n = i$ starting at $X_0 = 0$ and

let $P_i(x) = \sum_{n=0}^{\infty} p(n,i) x^n$ and $P(x,y) = \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} p(n,i) x^n y^i$.

Note that

$$p(n+1, i) = p(n, i-1) \mathbb{1}\{i>0\} + p(n, i+1), \quad n>0, \ i \geq 0$$

$$p(1,1) = p(0,0) = 1$$

Thus, $\frac{1}{x} \sum_{n=0}^{\infty} p(n+1, i) x^{n+1} = \sum_{n=0}^{\infty} p(n, i-1) \mathbb{1}\{i>0\} x^n + \sum_{n=0}^{\infty} p(n, i+1) x^n \Rightarrow$

$$\frac{1}{x}\left(P_i(x) - p(0,i)\right) = P_{i-1}(x) \mathbb{1}\{i>0\} + P_{i+1}(x) \Rightarrow$$

$$P_i(x) = p(0,i) + x P_{i-1}(x) \mathbb{1}\{i>0\} + x P_{i+1}(x), \ i \geq 0 \qquad (1)$$

Eq. (1) yields

$$P(x,y) = p(0,0) + xy P(x,y) + \frac{x}{y}\left(P(x,y) - P(x,0)\right) \Rightarrow$$

$$\boxed{\left(y - xy^2 - x\right) P(x,y) = y - x P(x,0)} \qquad (2)$$

Step 1 : Choose the y roots of the kernel of Eq. (2), i.e.,

$$y_{\pm}(x) = \frac{-1 \pm \sqrt{1-4x^2}}{-2x}$$

Step 2 : Note that for $y_+(x) = \frac{-1+\sqrt{1-4x^2}}{-2x} = \frac{1-\sqrt{1-4x^2}}{2x}$, $P(x, y_+(x))$ is well defined (why?), thus yielding $xP(x,0) = y_+(x)$ from which the calculation of $P(x,y)$ follows evidently.

Solution : $P(x,y) = \frac{y - xP(x,0)}{y - xy^2 - x} = \frac{y - y_+(x)}{y - xy^2 - x}$, $|x||y| \le 1$.