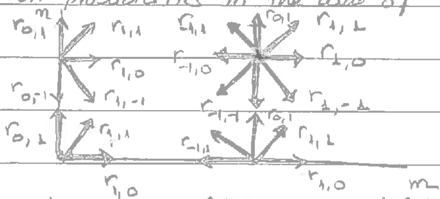
Generating function approach

Consider a nearest neighbor random walk restricted in the quadrant, with transition rates (OR probabilities in the case of discrete time) as depicted in the figure below



Let $\pi_{n,m}$ be the equilibrium probabilities, with $n, m \geq 0$, and $P(x,y) = \sum_n \sum_m \pi_{n,m} x^n y^m$, $|x|, |y| \leq 1$. Multiplying the balance equations with x^n and y^m, after summing for $n, m \geq 0$ yields a key functional equation for the unknown $P(x,y)$, namely

$$K(x,y)\, P(x,y) = A(x,y)\, P(x,0) + B(x,y)\, P(0,y) + C(x,y)\, P(0,0) \tag{1}$$

where $K(x,y)$, $A(x,y)$, $B(x,y)$, $C(x,y)$ are known polynomials depending on the parameters of the system. Equations of this type have been studied in the monographs [2] and [3]. The approach suggested consists of mainly two steps :

Step 1 : Consider the zeros of the kernel $K(x,y) = 0$ with $|x|, |y| \leq 1$. For such pairs $P(x,y)$ is analytic, and hence, for these pairs the right hand side of (1) is equal to zero.

Step 2 : For the pairs (x,y) satisfying Step 1, one needs to translate the fact that the right hand side of (1) is equal to zero into a Riemann(-Hilbert) boundary value problem.

Unfortunately, the above steps do not constitute a simple, straightforward recipe. In order to illustrate the procedure we consider the 2×2 switch.

Intermezzo

Let X a discrete r.v. with pdf $(p_n)_{n\geq 0}$, then the probability generating function (PGF) is defined as $P(z) = \sum_{n=0}^{\infty} p_n z^n = \mathbb{E}(z^X)$, $|z| \leq 1$.

Properties

i) $P(1) = 1$

ii) $p_n = \dfrac{P^{(n)}(0)}{n!}$

iii) $\mathbb{E}\left(\dfrac{X!}{(X-n)!}\right) = P^{(n)}(1)$

iv) $P(e^z) = M(z) = \mathbb{E}(e^{zX})$ which is the moment generating function

v) Setting $z := z+1$, yields

$$P(z+1) = \mathbb{E}\left((z+1)^X\right) = \mathbb{E}\left(\sum_{k=0}^{\infty} \binom{X}{k} z^k\right) \left(= \sum_{n=0}^{\infty} p_n \sum_{k=0}^{n} \binom{n}{k} z^k = \sum_{k=0}^{\infty} z^k \sum_{n=k}^{\infty} p_n \binom{n}{k} \right)$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{k!} m_{(k)}, \quad \text{with } m_{(k)} = \mathbb{E}[X(X-1)\cdots(X-k+1)]$$

The latter is known as the factorial moment generating function.

e.g. M/M/1 queue with $P(z) = \frac{1-\rho}{1-\rho z}$, $|z| < \frac{1}{\rho}$. Then

$$P(z+1) = \frac{1-\rho}{1-\rho-\rho z} = \frac{1}{1-\frac{\rho}{1-\rho}z} = \sum_{k=0}^{\infty} \left(\frac{\rho}{1-\rho}\right)^k z^k$$

$$\Rightarrow m_{(k)} = \left(\frac{\rho}{1-\rho}\right)^k k!, \quad k=0,1,\ldots$$

This last equation produces all moments, e.g. $\mathbb{E}(X) = m_{(1)} = \frac{\rho}{1-\rho}$,
$\mathbb{E}(X^2) = \mathbb{E}(X(X-1)) + \mathbb{E}(X) = m_{(2)} + m_{(1)}$, etc.
Note that the factorial moment generating function converges in $|z| < \frac{1-\rho}{\rho}$.

vi) Reconstruction of the distribution from all the moments.

Only, under certain conditions, e.g. if all moments exist and the moment generating function exists in some neighborhood of 0, cf. [page 65, 5]

Counter-example violating conditions

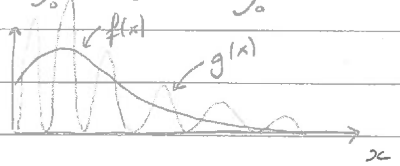
$$X \sim f(x) = \frac{1}{\sqrt{2\pi}\,x} \exp\left\{-\frac{1}{2}\ln^2(x)\right\}, \quad x>0$$

with $\mathbb{E}(X^n) = e^{\frac{1}{2}n^2}$, $n \in \mathbb{N}$ and $\mathbb{E}(e^{tX}) = \infty$, $t>0$

$$Y \sim g(x) = f(x)\,[1 + \sin(2\pi \ln x)], \quad x>0$$

with $\mathbb{E}(Y^n) = e^{\frac{1}{2}n^2}$, $n \in \mathbb{N}$ and g is a pdf.

Furthermore, $\int_0^\infty x^n g(x)\,dx = \int_0^\infty x^n f(x)\,dx$, however $g(x) \neq f(x)$



v) Numerical inversion of PGFs, see, eg [4], to find p_n

a) by inspection : p_n = coefficient of z^n term in $P(z)$

b) using Taylor expansion around $z=0$: $p_n = \frac{1}{n!} P^{(n)}(0)$

c) using Cauchy's residue theorem (contour integral in the complex plane):

$$p_n = \frac{1}{2\pi i} \oint \frac{P(z)}{z^{n+1}}\,dz$$

d) using partial fraction method :

$$P(z) = \frac{a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0}{b_d z^d + b_{d-1} z^{d-1} + \cdots + b_1 z + b_0} = \frac{N(z)}{D(z)}$$

$$= \sum_{k=0}^{m-d} c_k z^k + \frac{\tilde{N}(z)}{D(z)}$$

Let $D(z) = b_d (z-z_1)\cdots(z-z_d)$ if $D(z)$ has d distinct roots

Then

$$p_n \sim f_1 \alpha_1^{n+1}$$

with $\alpha_1 = \frac{1}{z_1} > \frac{1}{z_i} \ \forall i \neq 1$ and $f_1 = -\frac{\tilde{N}(z_1)}{D'(z_1)}$.

If the dominant root (z_1) has multiplicity 2, then $p_n \sim f_{1,2}\, n\, \alpha_1^{n+2}$

with $f_{1,2} = \frac{(-1)^2\, 2!\, \tilde{N}(z_1)}{D^{(2)}(z_1)}$.

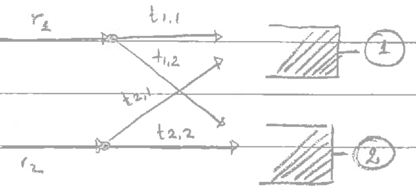
This can be generalized to any multiplicity of the dominant root.

e) Recursion method

$$P(z) = \frac{N(z)}{D(z)} = \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{d} p_k b_j z^{k+j}}{\sum_{j=0}^{d} b_j z^j} \quad \text{with} \quad \sum_{j=0}^{\min\{d,i\}} p_{i-j}\, b_j = \begin{cases} a_i, & i \leq n \\ 0, & i > n \end{cases}$$

Then, $p_n = \frac{1}{b_0}\left[a_n - \sum_{j=1}^{\min\{d,n\}} b_j\, p_{n-j}\right]$, $n \geq 0$

Boundary value method example : 2x2 switch, cf. [6]



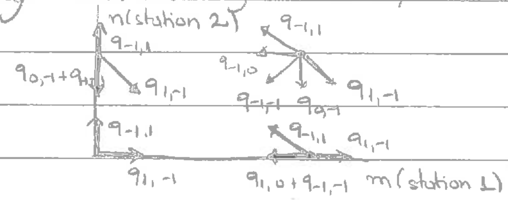
Consider the 2×2 clocked buffered switch with 2 input and 2 output ports. Modelling this as a DTMC :

r_i = prob. to have an arrival of type i, with $r_1 = r_2 = p$

t_{ij} = prob. a job of type i joins server j, with $t_{i,j} = \frac{1}{2}$

a server serves exactly one job per time unit

The probability transition diagram is depicted in the following figure.



with $q_{1,-1} = r_1 r_2 t_{1,1} t_{2,1}$, $q_{0,-1} = r_1(1-r_2)t_{1,1} + r_2(1-r_1)t_{2,1}$, $q_{-1,0} = r_1(1-r_2)t_{1,2} + r_2(1-r_1)t_{2,2}$

$q_{0,0} = r_1 r_2 (t_{1,1} t_{2,2} + t_{12} t_{2,1})$, $q_{-1,1} = r_1 r_2 \, t_{1,2} t_{2,2}$, $q_{-1,-1} = (1-r_1)(1-r_2)$

Let $P(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p_{m,n} x^m y^n$, $|x|,|y| \le 1$ be the PGF and $p_{m,n}$ the bi-variate equilibrium probability of having m (& respectively n) customers in station 1 (& respectively 2).

From the balance equations, it is straight forward to show that $P(x,y)$ satisfies the following functional equation

$$\big(xy - r(x,y)\big) P(x,y) = (y-1)\, r(x,0)\, P(x,0) + (x-1)\, r(0,y)\, P(0,y) + (x-1)(y-1)\, r(0,0)\, P(0,0) , \qquad |x|,|y| \le 1 \qquad (2)$$

with $r(x,y) = \left(1 - p + \frac{p}{2}(x+y)\right)^2$.

Step 1: Consider the zeros of the kernel $xy - r(x,y) = 0$. This gives us a very rich choice of zeros. Because of the symmetry of the model, we chose $y = \bar{x}$ (conjugate), then, the zeros of the kernel reduce to $|x|^2 - r(x,\bar{x}) = 0$ and let

$$E = \{ x \in \mathbb{C} : |x|^2 - r(x,\bar{x}) = 0, \ |x| \le 1 \}$$

Note that E forms an ellipse.

Step 2: For $y = \bar{x}$ and $x \in E$, Equation (2) assumes the form

$$0 = (\bar{x}-1)\, r(x,0)\, P(x,0) + (x-1)\, \overline{r(0,x)\, P(0,x)} + (x-1)(\bar{x}-1)\, r(0,0)\, P(0,0) \overset{x \neq 1}{\Longrightarrow}$$

$$0 = \frac{r(x,0)\, P(x,0)}{x-1} + \frac{\overline{r(0,x)\, P(0,x)}}{\bar{x}-1} + r(0,0)\, P(0,0) , \quad x \in E \setminus \{1\} \qquad (3)$$

Let $g(x) = \frac{r(x,0)\, P(x,0)}{x-1} + \frac{1}{2} r(0,0)\, P(0,0)$, then from (3), together with $P(x,0) = P(0,x)$

$$g(x) + g(\bar{x}) = 0 \Rightarrow 2\,\mathrm{Re}\, g(x) = 0 , \quad x \in E \setminus \{1\}$$

So, function $g(x)$ is analytic inside E, has a simple pole at 1, since

$$\lim_{x\to 1}(x-1)g(x) = \lim_{x\to 1} r(x,0)P(x,0) = r(1,0)P(1,0),$$

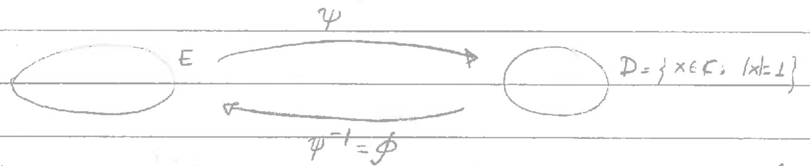
and on the boundary of E it has zero real part. So the problem reduces in identifying such a function $g(x)$.

If we would be looking at the unit circle, say D, then the problem of identifying the function is trivial; it can be easily verified that

$$h(\omega) = c\,\frac{\omega+1}{\omega-1}, \quad |\omega|<1 \qquad (4)$$

has a simple pole at 1 and on the boundary $\operatorname{Re} h(\omega)=0$ for $\omega\neq 1$ and $|\omega|=1$.

Now consider a function that maps the ellipse into the unit circle, say ψ.



that is $\psi: E \to D$, so $\omega = \psi(x)$. Then, (4) yields

$$h(\psi(x)) = c\,\frac{\psi(x)+1}{\psi(x)-1}$$

if, on top of that, we choose $\displaystyle\lim_{\omega\to 1}(\omega-1)h(\omega) \overset{\text{because of the mapping } \psi(1)=1}{=} \lim_{x\to 1}(\psi(x)-1)g(x) \Rightarrow 2c = r(1,0)P(1,0)\psi'(1)$

then $$g(x) = \frac{r(1,0)P(1,0)}{2}\,\frac{\psi(x)+1}{\psi(x)-1}\,\psi'(1) \qquad (5)$$

$$(2) \overset{(5)}{\Rightarrow} (xy - r(x,y))P(x,y) = (x-1)(y-1)\left[g(x)+g(y)\right] = \frac{r(1,0)P(1,0)}{2}\psi'(1)(x-1)(y-1)\left[\frac{\psi(x)+1}{\psi(x)-1} + \frac{\psi(y)+1}{\psi(y)-1}\right]$$

$$= \frac{r(1,0)P(1,0)}{\cancel{2}}(x-1)(y-1)\,\frac{\cancel{2}(\psi(x)\psi(y)-1)}{(\psi(x)-1)(\psi(y)-1)}\,\psi'(1)$$

$$\Rightarrow P(x,y) = r(1,0)P(1,0)\,\frac{(\psi(x)\psi(y)-1)}{xy-r(x,y)}\,\frac{(x-1)(y-1)}{(\psi(x)-1)(\psi(y)-1)}\,\psi'(1).$$

Lastly we show that $r(1,0)P(1,0) = 1-p$. Note that for $x=1$, (2) yields

$$(y-r(1,y))P(1,y) = (y-1)r(1,0)P(1,0) \Rightarrow P(1,y) = r(1,0)P(1,0)\,\frac{y-1}{y-r(1,y)} \Rightarrow$$

$$P(1,y) = r(1,0)P(1,0)\,\frac{y-1}{y-(1-p+py)^2} = r(1,0)P(1,0)\,\frac{1}{-p^2y+1-2p+p^2}$$

$$\overset{y=1}{\Rightarrow} 1 = r(1,0)P(1,0)\,\frac{1}{-p^2+1-2p+p^2} = r(1,0)P(1,0)\,\frac{1}{1-2p}$$

Furthermore $$P(1,y) = \frac{(1-2p)}{(1-p)^2}\,\frac{1}{1-\frac{p^2}{(1-p)^2}y} = \frac{1-2p}{(1-p)^2}\sum_{n=0}^{\infty}\left(p/_{1-p}\right)^{2n}y^n$$

References

[1] DP Kroese, WRW Scheinhart, and PG Taylor (2004). Spectral properties of the tandem Jackson network, seen as a QBD process. Ann. Appl. Prob. 14(4), 2057-89

[2] JW Cohen, and OJ Boxma (2000). Boundary Value Problems in Queueing System Analysis, volume 79. Elsevier.

[3] G Fayolle, R Iasnogorodski, and V. Malyshev (1999). Random Walks in the Quarter-Plane. Springer.

[4] J Abate, and W. Whitt (1992). Numerical inversion of PGFs. Operations Research Letters 12, 245-251.

[5] G Casella, and RL Berger (1990) Statistical Inference. Duxbury Press

[6] IJBF Adan, OJ Boxma, and JAC Resing (2001). Queueing models with multiple waiting lines. Queueing Systems 37(1), 65-98.

Generating function approach (continued)

The boundary value method approach discussed on page 5 relies on

1) being able to chose an "appropriate" set of zeros of the kernel (Step 1).
2) For the zeros choosen in Step 1 on page 5, we need to show that the right hand side of (1) is analytic in the interior of E, is continuous in E and then formulate a solvable boundary value problem, e.g. Hilbert Riemann-Hilbert, Dirichlet, etc.
3) Typically, the solvable boundary value problems are defined on the unit circle. Thus we need to obtain an expression for the mapping $\varphi : E \to D$. For smooth (bounded) contour one may employ Theodorsen's procedure, see [Section I.4 and Section IV.1.3, 2].

The combination of the above ingredients makes this approach technically challenging and its successful application cannot be guaranteed.

We will illustrate in the sequel two approaches that can be used, under a certain structure, to solve the functional equation (1).

Uniformization method

The global idea of the uniformization method is as follows: Uniformize the kernel equation $K(x,y)=0$ by introducing a uniformizing variable, say p, by writing

$$x := x(p) \quad \text{and} \quad y := y(p).$$

Consider $P(x(p),0)$ and $P(0,y(p))$ and show they are analytic in certain p-regions. Furthermore, for p such that $|x(p)|, |y(p)| < 1$ and $K(x(p),y(p))=0$, solve

$$A(x(p),y(p))\, K(x(p),0) + B(x(p),y(p))\, K(0,y(p)) + C(x(p),y(p))\, K(0,0) = 0.$$

Uniformization method example : 2×2 switch

Consider the 2×2 switch model as introduced in the previous lecture. The objective here is to solve the functional equation (2) using a uniformizing variable, p.

Step 1: Let $x := p$ and $y := y(p)$ such that $p\,y(p) - r(p, y(p)) = 0$. More concretely, note that the kernel $K(x,y) = xy - r(x,y)$ is a quadratic polynomial in both x and y. Thus, for a fixed $x := p$, there are two zero roots in y, say $y_1(p)$ and $y_2(p)$, with

$$|y_2(p)| < |p| < |y_1(p)| \quad , \text{ for } |p| \geq 1 \text{ and } p \neq 1$$

Step 2: For $x := p$ and $y := y_1(p)$, Equation (2) reduces to

$$0 = (y_1(p) - 1)\, r(p,0)\, P(p,0) + (p-1)\, r(0, y_1(p))\, P(0, y_1(p)) + $$
$$+ (p-1)(y_1(p)-1)\, r(0,0)\, P(0,0) \Rightarrow$$
$$-(p-1)\, r(0, y_1(p))\, P(0, y_1(p)) = (y_1(p)-1)\, r(p,0)\, P(p,0) + (p-1)(y_1(p)-1)\, r(0,0)\, P(0,0). \quad (6)$$

Note that for $p = 1$ the right hand side of (6) is obviously finite and $y_1(1) = \left(\frac{2-p}{p}\right)^2 > 1 \left(y_2(1) = 1\right)$. So $y_1(1)$ is a simple pole of $P(0, y_1(p))$. Hence, $P(0,y)$ can be analytically continued out from $|y| \leq 1$ into $\{y : 1 \leq |y| < y_1(1)\}$.

Performing the same analysis as before for the uniformizing variable $y := q$ and $x = x(q)$ such that $x(q)\,q - r(x(q), q) = 0$, with

$$|x_2(q)| < |q| < |x_1(q)| \quad \text{for } |q| \geq 1 \text{ and } q \neq 1.$$

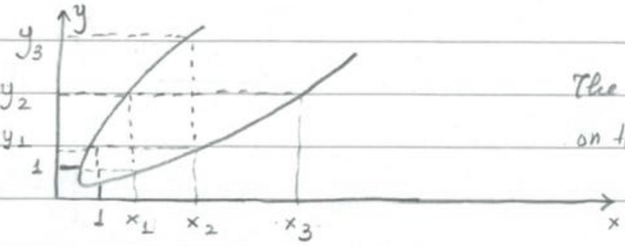
Furthermore, Equation (2) yields

$$0 = (q-1)\, r(x_1(q), 0)\, P(x_1(q), 0) + (x_1(q)-1)\, r(0,q)\, P(0,q) + (x_1(q)-1)(q-1)\, r(0,0)\, P(0,0) \Rightarrow$$
$$-(q-1)\, r(x_1(q),0)\, P(x_1(q),0) = (x_1(q)-1)\, r(0,q)\, P(0,q) + (x_1(q)-1)(q-1)\, r(0,0)\, P(0,0) \quad (7)$$

Note that for $q = 1$ the right hand side of (7) is bounded and thus $P(x,0)$ can be analytically continued out of $|x| \leq 1$ into $\{x : 1 \leq |x| < x_1(1)\}$.

This idea can be recursively applied back to (6)-(7) to further extend the analyticity of both $P(x,0)$ and $P(0,y)$ besides at the points in which these functions have simple poles. Note that in every finite domain in $\mathbb{C}$ the functions $P(x,0)$ and $P(0,y)$ have a finite number of poles $\Rightarrow$ $P(x,0)$ and $P(0,y)$ are meromorphic.

Say $y_1 = y_1(1)$, $x_1 = x_1(1)$, $y_n = y_1(x_{n-1}) = x_n = x_1(y_{n-1})$, $n = 2, 3, \ldots$ the sequence of simple poles we have obtained.



The sequences of simple poles $\{x_n\}_{n\geq 1}$, $\{(y_n)\}_{n\geq 1}$ on the parabola defined by $xy - r(x,y) = 0$.

Now for $p \in \mathbb{C} \setminus \{(x_n)\}_{n\geq 1}$, Equation (6) can be written as

$$P(p,0) = -\frac{p-1}{y_1(p)-1}\,\frac{r(0,y_1(p))}{r(p,0)}\,P(0,y_1(p)) - (p-1)\,\frac{r(0,0)}{r(p,0)}\,P(0,0) \qquad (8)$$

Note that there are p such that $r(p,0)=0$, thus $P(0,y_1(p))=0$. We can easily see that $r(p,0)=0 \Rightarrow p = 2(1-\frac{1}{p})$ "is a double root". Let $y_1^* = y_1(2(1-\frac{1}{p}))$. Similarly to before we define the sequences $\{(x_n^*)\}_{n\geq 1}$, $\{(y_n^*)\}_{n\geq 1}$ for which $P(x_n^*,0)=0$ and $P(0,y_n^*)=0$, and x_n^*, y_n^* are double roots.

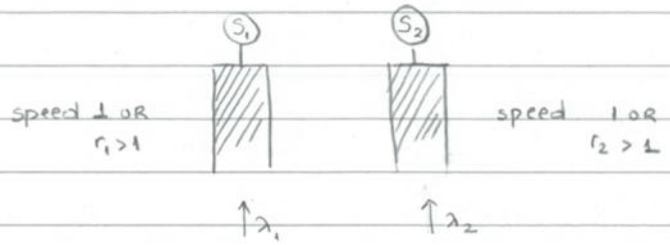
Based on all above information

$$P(x,0) = P(1,0)\,\frac{\prod_{n=1}^{\infty}\left(1-\frac{1}{x_n}\right)}{\prod_{n=1}^{\infty}\left(1-\frac{x}{x_n}\right)}\;\frac{\prod_{n=1}^{\infty}\left(1-\frac{x}{x_n^*}\right)^2}{\prod_{n=1}^{\infty}\left(1-\frac{1}{x_n^*}\right)^2}, \quad x \in \mathbb{C}.$$

The expression for $P(1,0)$ was obtained on page 6.

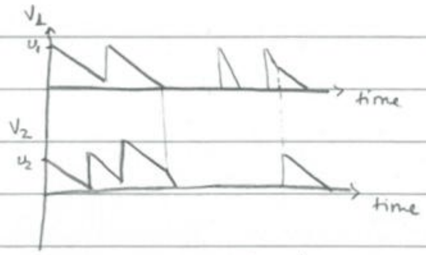
Similarly, we obtain $P(0,y)$, from that we can reconstruct $P(x,y)$.

Uniformization method example : Coupled processors.



Consider two M/G/1 queues with arrival rates λ_1 and λ_2, general service distribution B_1, and B_2, coupled via speeds. The speed of server i is 1 if the other queue is not empty and r_i if the other queue is empty, $r_i > 1$, $i = 1, 2$.

Let V_i denote the workload at processor i, $i = 1, 2$.



Let $\varphi(s_1, s_2) = \mathbb{E}\left[e^{-s_1 V_1 - s V_2}\right]$, then we can produce the functional equation of the bi-variate Laplace transform, which reads as follows

$$\underbrace{\left(\lambda_1 (1 - \tilde{B}_1(s_1)) - s_1 + \lambda_2 (1 - \tilde{B}_2(s_2)) - s_2\right)}_{K(s_1, s_2)} \Psi(s_1, s_2) = -\left((r_1 - 1)s_1 + (r_2 - 1) s_2\right) \Psi(0,0) + \left((r_1 - 1) s_1 - s_2\right) \Psi(s_1, 0) + \left((r_2 - 1) s_2 - s_1\right) \Psi(0, s_2) \qquad (9)$$

with $\tilde{B}_i(s_i) = \mathbb{E}\left(e^{-s_i B_i}\right)$, $i = 1, 2$, and $s_i \in \mathbb{C}_+$ (i.e. $\operatorname{Re} s_i \geq 0$).

For more details on the model and the derivations for Equation (9) the interested reader is referred to **[2]**, part III, Chapter III.3

Based on the approach we discussed in the previous lecture we can undertake two steps :

Step 1) Consider the zeros of the kernel $K(s_1, s_2) = 0$

Step 2) Solve the "boundary value problem".

For Step 1, notice that the kernel is seperable, i.e. $K(s_1, s_2) = f_1(s_1) + f(s_2)$, with $f_i(s_i) = \lambda_i (1 - \tilde{B}_i(s_i)) - s_i$. So

$$K(s_1, s_2) = f_1(s_1) + \omega + f_2(s_2) - \omega .$$

Uniformization method

Look for zeros $f_1(s_1)+\omega = f_2(s_2)-\omega = 0$. According to [7], under certain conditions $f_1(s_1)+\omega=0$ has exactly one zero $s_1 = \delta_1(\omega)$ for $Re\,\omega \geq 0$, $\omega \neq 0$, $Re\, s_1 \geq 0$, with multiplicity one. Furthermore $\delta_1(\omega)$ is analytic in $Re\,\omega > 0$ and continuous in $Re\,\omega \geq 0$.

Similarly $f_1(s_1)-\omega=0$ has exactly one zero $s_2 = \delta_2(\omega)$ for $Re\,\omega \leq 0$, $\omega \neq 0$, $Re\, s_1 \geq 0$ with multiplicity one. Furthermore $\delta_2(\omega)$ is analytic in $Re\,\omega < 0$ and continuous in $Re\,\omega \leq 0$.

Setting $s_i = \delta_i(\omega)$ in Equation (9) yields

$$0 = -\left((r_1-1)\delta_1(\omega) + (r_2-1)\delta_2(\omega)\right)\Psi(0,0) + \left((r_1-1)\delta_1(\omega) - \delta_2(\omega)\right)\Psi(\delta_1(\omega),0) + \\ + \left((r_2-1)\delta_2(\omega) - \delta_1(\omega)\right)\Psi(0,\delta_2(\omega)).$$

$$\Rightarrow \left(1-\frac{1}{r_1}\right)\delta_1(\omega) - \frac{1}{r_1}\delta_2(\omega)\left[\frac{1}{r_2}\left(\Psi(\delta_1(\omega),0) - \Psi(0,0)\right) - \frac{\Psi(0,0)}{r_1 r_2}\frac{1}{1-\frac{1}{r_1}-\frac{1}{r_2}}\right] =$$

$$= \left(1-\frac{1}{r_2}\right)\delta_2(\omega) - \frac{1}{r_2}\delta_1(\omega)\left[\frac{1}{r_1}\left(\Psi(0,\delta_2(\omega)) - \Psi(0,0)\right) - \frac{\Psi(0,0)}{r_1 r_2}\frac{1}{1-\frac{1}{r_1}-\frac{1}{r_2}}\right] \qquad (10)$$

with $\frac{1}{r_1}+\frac{1}{r_2} \neq 1$.

A careful analysis, shows that the LHS of (10) is analytic in $Re\,\omega > 0$ and continuous in $Re\,\omega \geq 0$, while the RHS of (10) is analytic in $Re\,\omega < 0$ and continuous in $Re\,\omega \leq 0$.

Using now Liouville's Theorem, it is evident that both sides are constants in ω. Thus, this determines $\Psi(\delta_1(\omega),0)$ for $Re\,\omega \geq 0$ and $\Psi(0,\delta_2(\omega))$ for $Re\,\omega \leq 0$. Subsequently, we can determine $\Psi(s_1,0)$ for $Re\, s_1 \geq 0$ and $\Psi(0,s_2)$ for $Re\, s_2 \leq 0$. Finally, $\Psi(s_1,s_2)$ follows from (9).

References

[7] JW Cohen (1982) The Single Server Queue. North-Holland Publications.