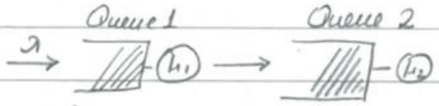
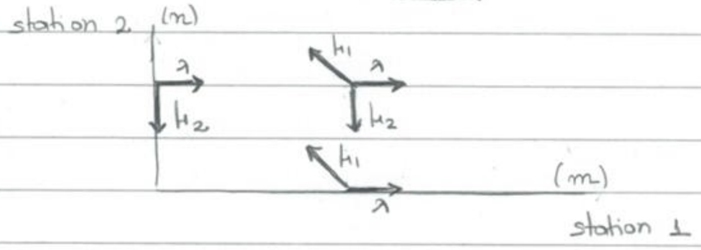
Question: Can we apply the theory of QBD's to processes with infinite phase-space?

Consider a Jackson network consisting of 2-stations and model it as a QBD.





The transition diagram for the tandem network

Let $\{(Y_t, J_t), t \geq 0\}$ be the stochastic process modeling the level (number of customers in station 2), Y_t, and the phase (number of customers in station 1), J_t, with state space $S = \{(n,m),\ n,m \in \mathbb{N}_0\}$. The corresponding generator matrix Q assumes the form

$$Q = \begin{bmatrix} \tilde{Q}_1 & Q_0 & & \\ Q_2 & Q_1 & Q_0 & \\ & \ddots & \ddots & \ddots \end{bmatrix}$$

with

$$Q_0 = \begin{bmatrix} 0 & & & \\ \mu_1 & 0 & & \\ & \mu_1 & 0 & \\ & & \ddots & \ddots \end{bmatrix}, \quad Q_2 = \mu_2 \mathbb{I}, \quad Q_1 = \begin{bmatrix} -(\lambda+\mu_2) & \lambda & & \\ & -(\lambda+\mu_1+\mu_2) & \lambda & \\ & & -(\lambda+\mu_1+\mu_2) & \ddots \\ & & & \ddots \end{bmatrix}$$

and $\tilde{Q}_1 = Q_1 + Q_2$.

We know that this process is stable if $\lambda < \min\{\mu_1, \mu_2\}$. Then, the equilibrium probabilities are $\pi_{n,m} = (1-\rho_1)\rho_1^n (1-\rho_2)\rho_2^m$, $n,m \geq 0$.

Consider now that the phase is truncated, at say value M. We then consider $\pi_n = (\pi_{n,0}, \pi_{n,1}, \dots, \pi_{n,M})$ for $n \geq 0$, and $\pi_n = \pi_0 R^n$, $n \geq 0$, with R_M the minimal non-negative solution to $Q_0 + R_M Q_1 + R_M^2 Q_2 = 0$. For $M < \infty$, the decay rate at level n of the QBD process is $sp(R_M)$ (spectral radius of R_M), i.e.

$$\lim_{n \to \infty} \frac{\sum_m \pi_{n,m}}{sp(R_M)^n} = c$$

with c some constant.

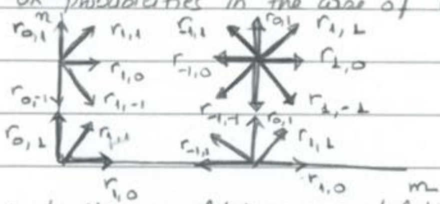
Then, it can be shown, cf. [Corollary 5.5, 1], that, if $\mu_1 < \mu_2$ then $sp(R_M)$ increases to η as M increases, while if $\mu_1 > \mu_2$ then $sp(R_M)$ increases to ρ_2 as M increases. With η the unique solution in $(0,1)$ of the function

$$-\lambda - \mu_1 - \mu_2(1-\eta) + 2\sqrt{\frac{\lambda\mu_1}{\eta}} = 0.$$

So, based on this example, we should avoid truncating the state-space of the process. Instead we will investigate other approaches for the calculation of the equilibrium probabilities. Furthermore, we will return to the use of QBD's in a future lecture.

Generating function approach

Consider a nearest neighbor random walk restricted in the quadrant, with transition rates (OR probabilities in the case of discrete time) as depicted in the figure below



Let $\pi_{n,m}$ be the equilibrium probabilities, with $n, m \geq 0$, and $P(x,y) = \sum_n \sum_m \pi_{n,m} x^n y^m$, $|x|, |y| \leq 1$. Multiplying the balance equations with x^n and y^m, after summing for $n, m \geq 0$, yields a key functional equation for the unknown $P(x,y)$, namely

$$K(x,y)\,P(x,y) = A(x,y)\,P(x,0) + B(x,y)\,P(0,y) + C(x,y)\,P(0,0) \qquad (1)$$

where $K(x,y)$, $A(x,y)$, $B(x,y)$, $C(x,y)$ are known polynomials depending on the parameters of the system. Equations of this type have been studied in the monographs [2] and [3]. The approach suggested consists of mainly two steps:

Step 1: Consider the zeros of the kernel $K(x,y) = 0$ with $|x|, |y| \leq 1$. For such pairs $P(x,y)$ is analytic, and hence, for these pairs the right hand side of (1) is equal to zero

Step 2: For the pairs (x,y) satisfying Step 1, one needs to translate the fact that the right hand side of (1) is equal to zero into a Riemann(-Hilbert) boundary value problem

Unfortunately, the above steps do not constitute a simple, straightforward recipe. In order to illustrate the procedure we consider the 2×2 switch.

Intermezzo

Let X a discrete r.v. with pdf $(p_n)_{n\geq 0}$, then the probability generating function (PGF) is defined as $P(z) = \sum_{n=0}^{\infty} p_n z^n = \mathbb{E}(z^X)$, $|z| \leq 1$.

Properties

i) $P(1) = 1$

ii) $p_n = \dfrac{P^{(n)}(0)}{n!}$

iii) $\mathbb{E}\left(\dfrac{X!}{(X-n)!}\right) = P^{(n)}(1)$

iv) $P(e^z) = M(z) = \mathbb{E}(e^{zX})$ which is the moment generating function

v) Setting $z := z+1$, yields

$$P(z+1) = \mathbb{E}((z+1)^X) = \mathbb{E}\left(\sum_{k=0}^{\infty} \binom{X}{k} z^k\right) \left(= \sum_{n=0}^{\infty} p_n \sum_{k=0}^{n} \binom{n}{k} z^k = \sum_{k=0}^{\infty} z^k \sum_{n=k}^{\infty} p_n \binom{n}{k} \right)$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{k!}\, m_{(k)}, \quad \text{with } m_{(k)} = \mathbb{E}[X(X-1)\cdots(X-k+1)]$$

The latter is known as the factorial moment generating function.

e.g. M/M/1 queue with $P(z) = \frac{1-\rho}{1-\rho z}$, $|z| < \frac{1}{\rho}$. Then

$$P(z+1) = \frac{1-\rho}{1-\rho-\rho z} = \frac{1}{1-\frac{\rho}{1-\rho}z} = \sum_{k=0}^{\infty} \left(\frac{\rho}{1-\rho}\right)^k z^k$$

$$\Rightarrow m_{(k)} = \left(\frac{\rho}{1-\rho}\right)^k k!\ , \quad k=0,1,\ldots$$

This last equation produces all moments, eg. $\mathbb{E}(X) = m_{(1)} = \frac{\rho}{1-\rho}$,
$\mathbb{E}(X^2) = \mathbb{E}(X(X-1)) + \mathbb{E}(X) = m_{(2)} + m_{(1)}$, etc.
Note that the factorial moment generating function converges in $|z| < \frac{1-\rho}{\rho}$.

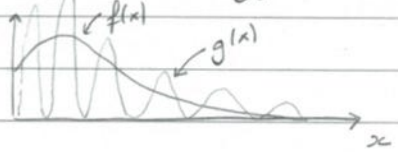
vi) Reconstruction of the distribution from all the moments.
Only, under certain conditions, eg if all moments exist and the moment generating function exists in some neighborhood of 0, cf. [page 65, 5]
Counter-example violating conditions

$$X \sim f(x) = \frac{1}{\sqrt{2\pi}\, x} \exp\left\{-\frac{1}{2}\ln^2(x)\right\}, \quad x>0$$

with $\mathbb{E}(X^n) = e^{\frac{1}{2}n^2}$, $n \in \mathbb{N}$ and $\mathbb{E}(e^{tX}) = \infty$, $t>0$.

$$Y \sim g(x) = f(x)\,[1 + \sin(2\pi \ln x)], \quad x>0$$

with $\mathbb{E}(Y^n) = e^{\frac{1}{2}n^2}$, $n \in \mathbb{N}$ and g is a pdf.
Furthermore, $\int_0^\infty x^n g(x)\,dx = \int_0^\infty x^n f(x)\,dx$, however $g(x) \neq f(x)$



v) Numerical inversion of PGFs, see, eg [4], to find p_n.
a) by inspection : p_n = coefficient of z^n term in $P(z)$
b) using Taylor expansion around $z=0$: $p_n = \frac{1}{n!} P^{(n)}(0)$
c) using Cauchy's residue theorem (contour integral in the complex plane):

$$p_n = \frac{1}{2\pi i} \oint \frac{P(z)}{z^{n+1}}\, dz$$

d) using partial fraction method :

$$P(z) = \frac{a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0}{b_d z^d + b_{d-1} z^{d-1} + \cdots + b_1 z + b_0} = \frac{N(z)}{D(z)}$$

$$= \sum_{k=0}^{m-d} c_k z^k + \frac{\tilde{N}(z)}{D(z)}$$

Let $D(z) = b_d (z-z_1)\cdots(z-z_d)$ if $D(z)$ has d distinct roots
Then

$$p_n \sim f_1 \alpha_1^{n+1}$$

with $\alpha_1 = \frac{1}{z_1} > \frac{1}{z_i} \ \forall i \neq 1$ and $f_1 = -\frac{\widetilde{N}(z_1)}{D'(z_1)}$.

If the dominant root (z_1) has multiplicity 2, then $p_n \sim f_{1,2} \, n \, \alpha_1^{n+2}$
with $f_{1,2} = \frac{(-1)^2 \, 2! \, \widetilde{N}(z_1)}{D^{(2)}(z_1)}$.

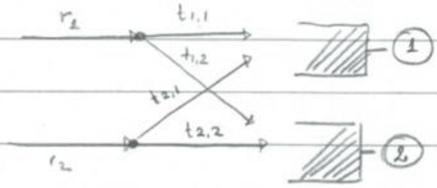
This can be generalized to any multiplicity of the dominant root.

e) Recursion method

$$P(z) = \frac{N(z)}{D(z)} = \frac{\sum_{k=0}^{\infty} \sum_{j=0}^{d} p_k \, b_j \, z^{k+j}}{\sum_{j=0}^{d} b_j z^j} \quad \text{with} \quad \sum_{j=0}^{\min\{d,i\}} p_{i-j} \, b_j = \begin{cases} a_i & , i \leq n \\ 0 & , i > n \end{cases}$$

Then, $p_n = \frac{1}{b_0} \left[a_n - \sum_{j=1}^{\min\{d,n\}} b_j \, p_{n-j} \right]$, $n \geq 0$.

Boundary value method example : 2x2 switch, cf. [6]



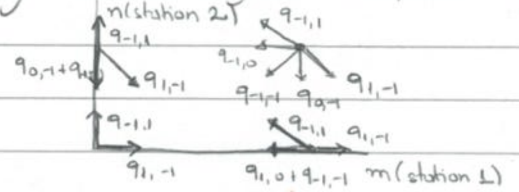
Consider the 2×2 clocked buffered switch with 2 input and 2 output ports. Modelling this as a DTMC :

r_i = prob. to have an arrival of type i, with $r_1 = r_2 = p$

t_{ij} = prob. a job of type i joins server j, with $t_{i,j} = \frac{1}{2}$

a server serves exactly one job per time unit

The probability transition diagram is depicted in the following figure.



with $q_{1,-1} = r_1 r_2 t_{1,1} t_{2,1}$, $q_{0,-1} = r_1(1-r_2)t_{1,1} + r_2(1-r_1)t_{2,1}$, $q_{-1,0} = r_1(1-r_2)t_{1,2} + r_2(1-r_1)t_{2,2}$

$q_{0,0} = r_1 r_2 (t_{1,1} t_{2,2} + t_{1,2} t_{2,1})$, $q_{-1,1} = r_1 r_2 t_{1,2} t_{2,2}$, $q_{-1,-1} = (1-r_1)(1-r_2)$

Let $P(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p_{m,n} x^m y^n$, $|x|,|y| \le 1$ be the PGF and $p_{m,n}$ the bi-variate equilibrium probability of having m (& respectively n) customers in station 1 (& respectively 2).
From the balance equations, it is straight forward to show that $P(x,y)$ satisfies the following functional equation

$$\left(xy - r(x,y)\right) P(x,y) = (y-1)\, r(x,0)\, P(x,0) + (x-1)\, r(0,y)\, P(0,y) + (x-1)(y-1)\, r(0,0)\, P(0,0). \qquad |x|,|y| \le 1 \qquad (2)$$

with $r(x,y) = \left(1 - p + \frac{p}{2}(x+y)\right)^2$.

Step 1: Consider the zeros of the kernel $xy - r(x,y) = 0$. This gives us a very rich choice of zeros. Because of the symmetry of the model, we chose $y = \bar{x}$ (conjugate), then the zeros of the kernel reduce to $|x|^2 - r(x,\bar{x}) = 0$ and let

$$E = \{x \in \mathbb{C} : |x|^2 - r(x,\bar{x}) = 0,\ |x| \le 1\}$$

Note that E forms an ellipse.

Step 2: For $y = \bar{x}$ and $x \in E$, Equation (2) assumes the form

$$0 = (\bar{x}-1)\, r(x,0)\, P(x,0) + (x-1)\, \overline{r(0,x)\, P(0,x)} + (x-1)(\bar{x}-1)\, r(0,0)\, P(0,0) \overset{x \neq 1}{\Longrightarrow}$$

$$0 = \frac{r(x,0)\, P(x,0)}{x-1} + \frac{\overline{r(0,x)\, P(0,x)}}{\bar{x}-1} + r(0,0)\, P(0,0) , \quad x \in E \setminus \{1\} \qquad (3)$$

Let $g(x) = \frac{r(x,0)\, P(x,0)}{x-1} + \frac{1}{2} r(0,0)\, P(0,0)$, then from (3), together with $P(x,0) = P(0,x)$

$$g(x) + g(\bar{x}) = 0 \Rightarrow 2\,\mathrm{Re}\, g(x) = 0 , \quad x \in E \setminus \{1\}$$

So, function $g(x)$ is analytic inside E, has a simple pole at 1, since

$$\lim_{x\to 1}(x-1)\,g(x) = \lim_{x\to 1} r(x,0)\,P(x,0) = r(1,0)\,P(1,0),$$

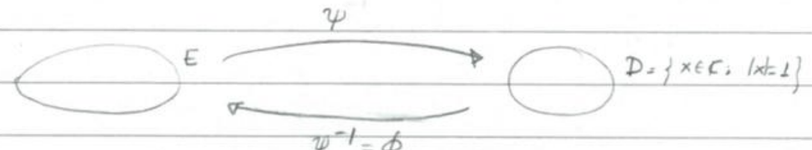
and on the boundary of E it has zero real part. So the problem reduces in identifying such a function $g(x)$.

If we would be looking at the unit circle, say D, then the problem of identifying the function is trivial; it can be easily verified that

$$h(w) = c\,\frac{w+1}{w-1}, \quad |w|<1 \tag{4}$$

has a simple pole at 1 and on the boundary $\operatorname{Re} h(w) = 0$ for $w\neq 1$ and $|w|=1$.

Now consider a function that maps the ellipse into the unit circle, say ψ,



that is $\psi : E \to D$, so $w = \psi(x)$. Then, (4) yields

$$h(\psi(x)) = c\,\frac{\psi(x)+1}{\psi(x)-1}$$

if, on top of that, we chose $\lim_{w\to 1}(w-1)\,h(w) \overset{\text{because of the mapping } \psi(1)=1}{=} \lim_{x\to 1}(\psi(x)-1)\,g(x) \Rightarrow 2c = r(1,0)\,P(1,0)\,\psi'(1)$

then $$g(x) = \frac{r(1,0)\,P(1,0)}{2}\,\frac{\psi(x)+1}{\psi(x)-1}\,\psi'(1) \tag{5}$$

$$(2) \overset{(5)}{\Rightarrow} (xy - r(x,y))P(x,y) = (x-1)(y-1)\left[g(x)+g(y)\right] = \frac{r(1,0)\,P(1,0)\,\psi'(1)}{2}(x-1)(y-1)\left[\frac{\psi(x)+1}{\psi(x)-1} + \frac{\psi(y)+1}{\psi(y)-1}\right]$$

$$= \frac{r(1,0)P(1,0)}{\cancel{2}}(x-1)(y-1)\,\frac{\cancel{2}\,(\psi(x)\psi(y)-1)}{(\psi(x)-1)(\psi(y)-1)}\,\psi'(1)$$

$$\Rightarrow P(x,y) = r(1,0)\,P(1,0)\,\frac{(\psi(x)\psi(y)-1)}{xy - r(x,y)}\,\frac{(x-1)(y-1)}{(\psi(x)-1)(\psi(y)-1)}\,\psi'(1).$$

Lastly we show that $r(1,0)P(1,0) = 1-p$. Note that for $x=1$, (2) yields

$$(y - r(1,y))P(1,y) = (y-1)\,r(1,0)\,P(1,0) \Rightarrow P(1,y) = r(1,0)\,P(1,0)\,\frac{y-1}{y-r(1,y)} \Rightarrow$$

$$P(1,y) = r(1,0)\,P(1,0)\,\frac{y-1}{y-\left(1-\frac{p}{2}+\frac{p}{2}y\right)^2} = r(1,0)\,P(1,0)\,\frac{1}{-\frac{p^2}{4}y + 1 - p + \frac{p^2}{4}}$$

$$\overset{y=1}{\Rightarrow} 1 = r(1,0)\,P(1,0)\,\frac{1}{-\frac{p^2}{4} + 1 - p + \frac{p^2}{4}} = r(1,0)\,P(1,0)\,\frac{1}{1-p} \Rightarrow r(1,0)\,P(1,0) = 1-p$$

Furthermore $$P(1,y) = \frac{(1-p)}{\left(1-\frac{p}{2}\right)^2}\,\frac{1}{1-\frac{p^2/4}{(1-p/2)^2}\,y} = \frac{1-p}{\left(1-\frac{p}{2}\right)^2}\sum_{n=0}^{\infty}\left(\frac{p/2}{1-p/2}\right)^{2n} y^n$$

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