Matrix geometric approach for random walks in the quadrant

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We use the known QBD stability condition


Let $A = A_{-1} + A_0 + A_1$ and $x$ be the unique solution to

$$ xA = 0 $$

such that $x \mathbf{1} = 1$, with $\mathbf{1}$ a column vector of ones.

Stable iff

$$ xA_1 \mathbf{1} < xA_{-1} \mathbf{1} $$

**Theorem**

The NNRW is stable iff

$$ x^H A_1^H \mathbf{1} < x^H A_{-1}^H \mathbf{1} $$

and

$$ x^V A_1^V \mathbf{1} < x^V A_{-1}^V \mathbf{1} $$
Example

Performance analysis of two coupled $M / M / 1$ queues (in parallel), where the coupling occurs due to simultaneous abandonments.

We transform the state space description:

$n = \min\{q_1, q_2\}$ and $m = q_2 - q_1$

Objective: stability condition?
Example

We transform the state space description
\[ n = \min\{q_1, q_2\} \text{ and } m = q_2 - q_1 \]

If \( \lambda_1 = \lambda_2 = \lambda \) and \( \mu_1 = \mu_2 = \mu \), then we transform the state space description
\[ n = \min\{q_1, q_2\} \text{ and } m = |q_2 - q_1| \]

**Objective:** stability condition?
Stability condition

This system is always stable.

If $\lambda_1 = \lambda_2 = \lambda$ and $\mu_1 = \mu_2 = \mu$, then we transform the state space description

$n = \min\{q_1, q_2\}$ and $m = |q_2 - q_1|$
Stability condition

\( \lambda_1 \neq \lambda_2 \) and \( \mu_1 \neq \mu_2 \)

We use the known QBD stability condition


Let \( A = A_{-1} + A_0 + A_1 \) and \( x \) be the unique solution to

\[ xA = 0 \]

such that \( x\bar{1} = 1 \), with \( \bar{1} \) a column vector of ones.

Stable iff

\[ xA_1\bar{1} < xA_{-1}\bar{1} \]

Conjecture

The stability condition is given by

\[ (1 - x_{0}^{up})\mu_1 + \lambda_2 < \lambda_1 + \mu_2 \]

and

\[ (1 - x_{0}^{lower})\mu_2 + \lambda_1 < \lambda_2 + \mu_1 \]
Exact analysis for random walks

- Boundary value method approach
  
  

\[ K(x,y)\Pi(x,y) = A(x,y)\Pi(x,0) + B(x,y)\Pi(0,y) + C(x,y)\Pi(0,0) \]

- Matrix geometric approach \( \pi_m = \pi_{m-1} R \)
  \[ A_1 + RA_0 + R^2 A_{-1} = 0 \]

- Compensation approach
  

- Successive lumping
  
Main results

We consider the class of nearest neighbour random walks (NNRW)

- 1st quadrant
- Homogeneous nearest neighbour
- No transitions to N, NE and E

![Diagram of NNRW](image)
Main results

We consider the class of nearest neighbour random walks (NNRW)
- 1st quadrant
- Homogeneous nearest neighbour
- No transitions to N, NE and E

Theorem 1
We consider the class of NNRW and we calculate the eigenvalues and eigenvectors of $R$ recursively.

Theorem 2
For the class of NNRW the infinite dimension rate matrix $R$ is “diagonalizable” and we can numerically approximate $R$ using spectral truncation.

Theorem 3
We obtain the eigenvalues of the rate matrix for the original model.
Nearest neighbour random walk

We consider the class of nearest neighbour random walks (NRRW):

- 1st quadrant
- Homogeneous nearest neighbour
- No transitions to N, NE and E

Then,

\[ \pi(m, n) \sim c \alpha^m \beta^n \quad \text{as } m, n \to \infty \]

More concretely,

\[ \pi(m, n) = \sum_i c_i \alpha_i^m \beta_i^n, \quad m, n > 0 \]

The limitations above are necessary

Boundary value method approach

First, introduce

\[ \Pi(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \pi(m, n)x^m y^n \]

then

\[ K(x, y)\Pi(x, y) = A(x, y)\Pi(x, 0) + B(x, y)\Pi(0, y) + C(x, y)\Pi(0, 0) \]

where \( K(x, y), A(x, y), B(x, y), C(x, y) \) are known quadratic functions.

Choose \( y = f(x) \), e.g. \( y = \bar{x} \), and set \( K(x, f(x)) = 0 \)

\[ 0 = A(x, f(x))\Pi(x, 0) + B(x, f(x))\Pi(0, f(x)) + C(x, f(x))\Pi(0, 0) \]

The above equation can be solved as a Riemann (Hilbert) boundary value problem.
Compensation approach

Aims at solving directly the balance equations of a random walk in the quadrant using a series (infinite or finite) of product-form solutions

Key idea:
- Guess a product-form solution $\alpha^m \beta^n$
- Check if it satisfies the boundaries
- If not start compensating by adding new product-form terms

Solution

$$\pi(m, n) = \sum_i c_i \alpha_i^m \beta_i^n, m, n > 0$$
Matrix geometric approach

We know that

\[ \pi_m = \pi_{m-1} R \]

where \( \pi_m = (\pi(m, 0) \ \pi(m, 1) \ \ldots) \) and \( \pi(m, n) = \sum_i c_i \alpha_i^m \beta_i^n \), \( m, n > 0 \).

Then,

\[ \Pi(x, y) = \pi_0 y' + \pi_1 (x^{-1} I - R)^{-1} y' \]

where \( y' = (1 \ y \ y^2 \ \ldots) \).

Substituting in the functional equation reveals

\[ K(x, y) \Pi(x, y) = A(x, y) \Pi(x, 0) + B(x, y) \Pi(0, y) + C(x, y) \Pi(0, 0) \Rightarrow \]

\[ \pi_1 (x^{-1} I - R)^{-1} [K(x, y)y + A(x, y)e] \]

\[ = -\pi_0 [(K(x, y) + B(x, y))y' + (A(x, y) + C(x, y))e'] \]

So \( x^{-1} = \alpha \) is an eigenvalue of matrix \( R \).

The terms \( y^{-1} = \beta \) are associated with the eigenvectors of \( R \).
Matrix geometric approach

Theorem 1
The terms \( \{\alpha_i\} \) constitute the different eigenvalues of the matrix \( R \). For eigenvalue \( \alpha_i \) the corresponding eigenvector of the matrix \( R \) is \( h_i \) with \( h_{i,n} = c_i (\beta^n_{i-1} + f_i \beta^n_i) \).

Theorem 2
Spectral decomposition

\[
R = H^{-1}DH
\]

Truncated spectral decomposition

\[
R_I = H_I^{-1}D_IH_I
\]

Remark
The latter is equivalent to truncating

\[
\pi(m, n) = \sum_{i=0}^{I} c_i \alpha_i^m \beta_i^n, m, n > 0
\]
Main results

Theorem 3
We obtain the eigenvalues of the rate matrix for the original model.

\[ K(x, y)\Pi(x, y) = A(x, y)\Pi(x, 0) + B(x, y)\Pi(0, y) + C(x, y)\Pi(0, 0) \]
\[ + D(x, y)\Pi(p + (1 - p)x, y) \]

By using a similar argument as previously we obtain

\[ \alpha_0 \rightarrow p + (1 - p)\alpha_0 \rightarrow p^2 + (1 - p^2)\alpha_0 \]

\[ \alpha_1 \rightarrow \vdots \rightarrow \vdots \]
Conclusions

- Calculation of eigenvalues and eigenvectors of rate matrix for NNRW
- Efficient numerical calculation of rate matrix using spectral truncation
- Our results show promise for “non-structured” $R$ of random walks in the quadrant

Extensions

- Probabilistic interpretation of the product-form terms
- Use the results for approximation, i.e. approximate the invariant measure by a series (finite or infinite) of product forms.