NET WORKS

TU/e Technische Universiteit Eindhoven University of Technology

Matrix geometric approach for random walks in the quadrant

Stella Kapodistria

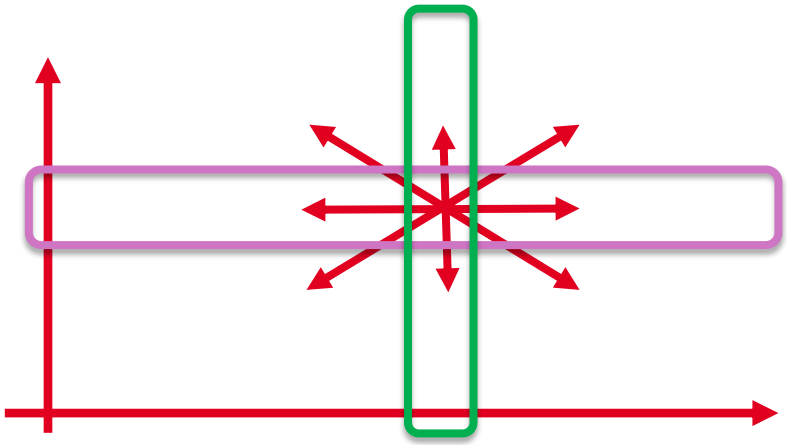
Joint work with
Zbigniew Palmowski and
Seva Shneer

Feb 11, 2019

Where innovation starts

Stability condition

Nearest neighbor random walk (NNRW)



We use the known QBD stability condition

[1] M.F. Neuts (1981). Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach.

Let $A = A_{-1} + A_0 + A_1$ and x be the unique solution to

$$xA = 0$$

such that $x\vec{1} = 1$, with $\vec{1}$ a column vector of ones.

Stable iff

$$xA_1\vec{1} < xA_{-1}\vec{1}$$

Theorem

The NNRW is stable iff

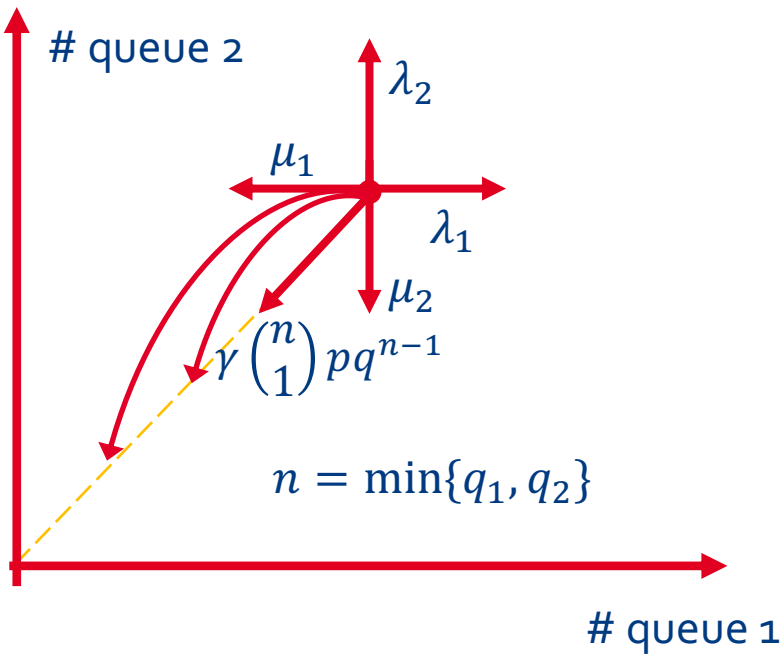
$$x^H A_1^H \vec{1} < x^H A_{-1}^H \vec{1}$$

$$\& \quad x^V A_1^V \vec{1} < x^V A_{-1}^V \vec{1}$$

Example

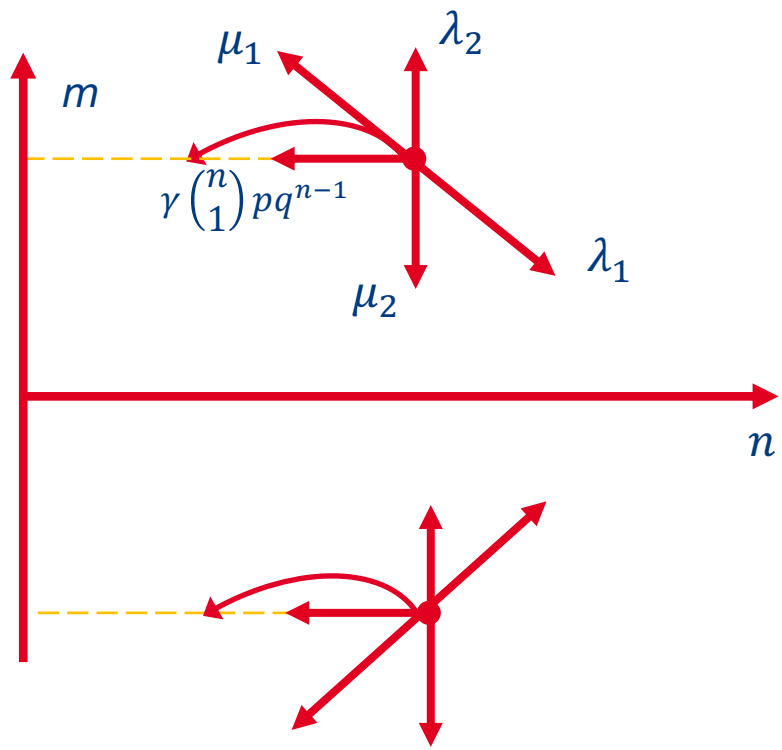
Performance analysis of

two coupled M / M /1 queues (in parallel), where the coupling occurs due to simultaneous abandonments



We transform the state space description

$n = \min\{q_1, q_2\}$ and $m = q_2 - q_1$

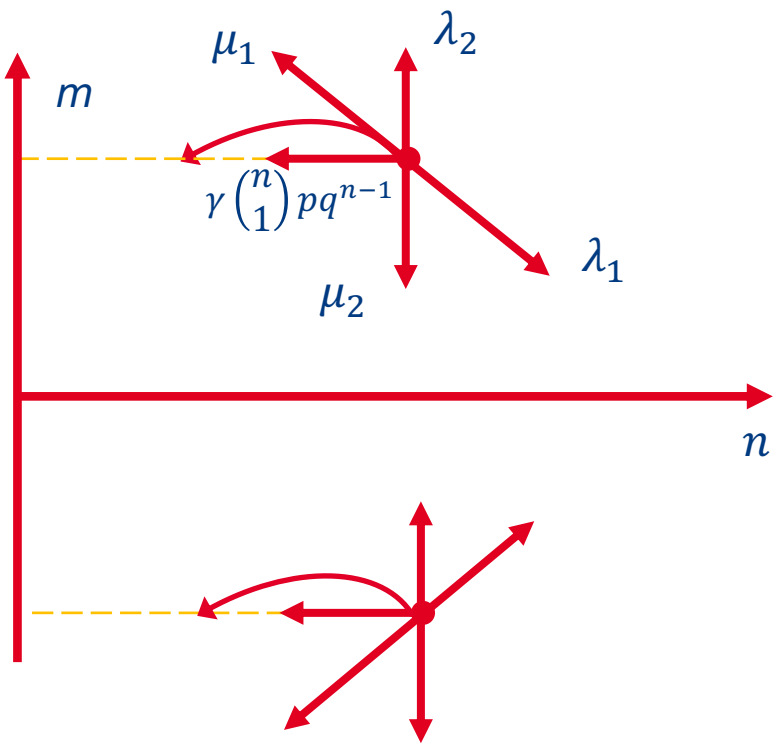


Objective: stability condition?

Example

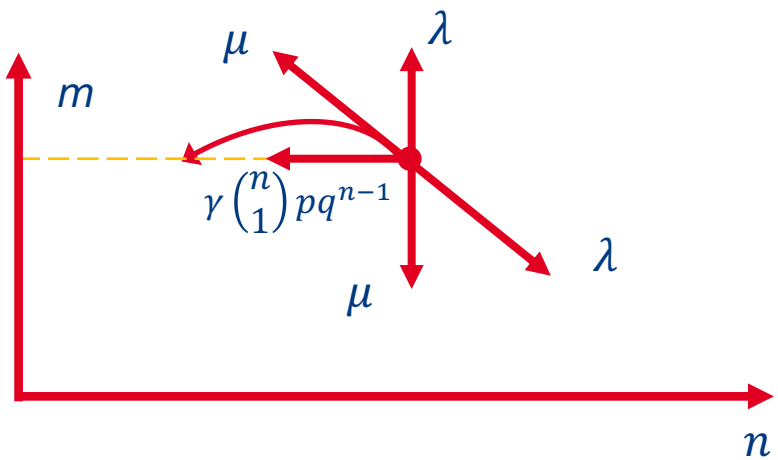
We transform the state space description

$$n = \min\{q_1, q_2\} \text{ and } m = q_2 - q_1$$



If $\lambda_1 = \lambda_2 = \lambda$ and $\mu_1 = \mu_2 = \mu$, then we transform the state space description

$$n = \min\{q_1, q_2\} \text{ and } m = |\, q_2 - q_1 \,|$$



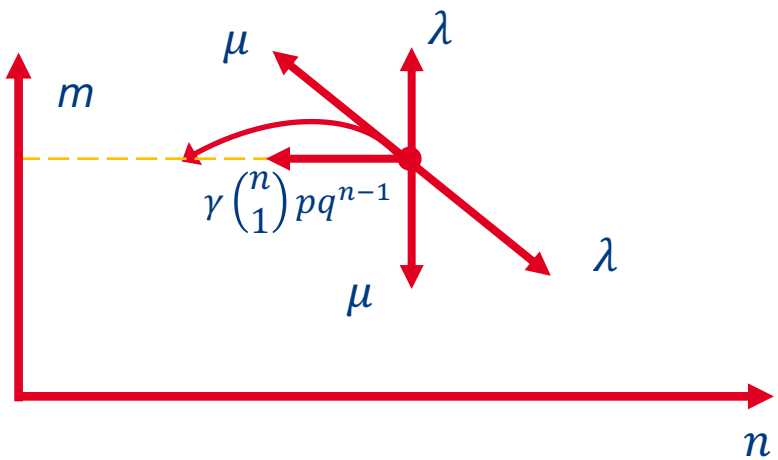
Objective: stability condition?

Stability condition

This system is always stable.

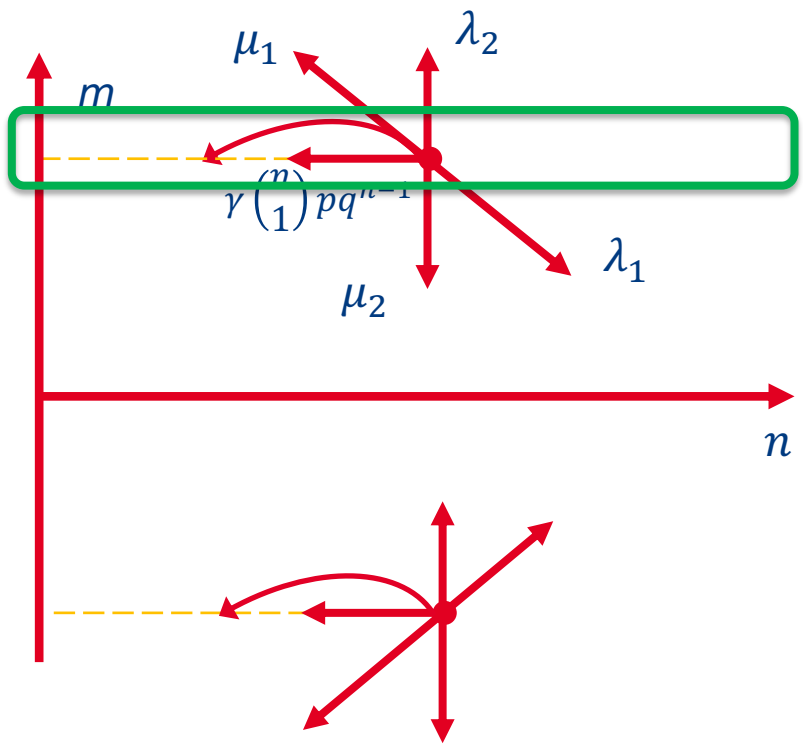
If $\lambda_1 = \lambda_2 = \lambda$ and $\mu_1 = \mu_2 = \mu$, then we transform the state space description

$$n = \min\{q_1, q_2\} \text{ and } m = |\, q_2 - q_1 \,|$$



Stability condition

$\lambda_1 \neq \lambda_2$ and $\mu_1 \neq \mu_2$



We use the known QBD stability condition

[1] M.F. Neuts (1981). Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach.

Let $A = A_{-1} + A_0 + A_1$ and x be the unique solution to

$$xA = 0$$

such that $x\vec{1} = 1$, with $\vec{1}$ a column vector of ones.

Stable iff

$$xA_1\vec{1} < xA_{-1}\vec{1}$$

Conjecture

The stability condition is given by

$$(1 - x_0^{up})\mu_1 + \lambda_2 < \lambda_1 + \mu_2$$

and

$$(1 - x_0^{lower})\mu_2 + \lambda_1 < \lambda_2 + \mu_1$$

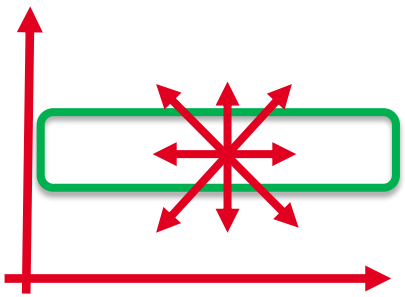
Exact analysis for random walks

- Boundary value method approach

  [2] Cohen, J.W. and Boxma, O.J. (1983). Boundary Value Problems in Queueing System Analysis.

  [3] Fayolle, G., Iasnogorodski, R. and Malyshev, V. (1999). Random Walks in the Quarter Plane.

$$K(x,y)\Pi(x,y) = A(x,y)\Pi(x,0) + B(x,y)\Pi(0,y) + C(x,y)\Pi(0,0)$$



- Matrix geometric approach $\boldsymbol{\pi}_m = \boldsymbol{\pi}_{m-1}\boldsymbol{R}$

$$\boldsymbol{A}_1 + \boldsymbol{R}\boldsymbol{A}_0 + \boldsymbol{R}^2\boldsymbol{A}_{-1} = \boldsymbol{0}$$

- Compensation approach

  [4] Adan, I.J.B.F. (1991). A Compensation Approach for Queueing Problems.

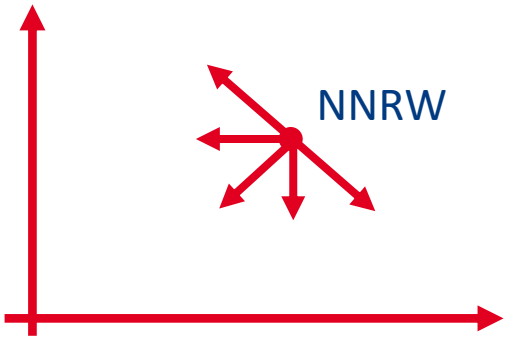
- Successive lumping

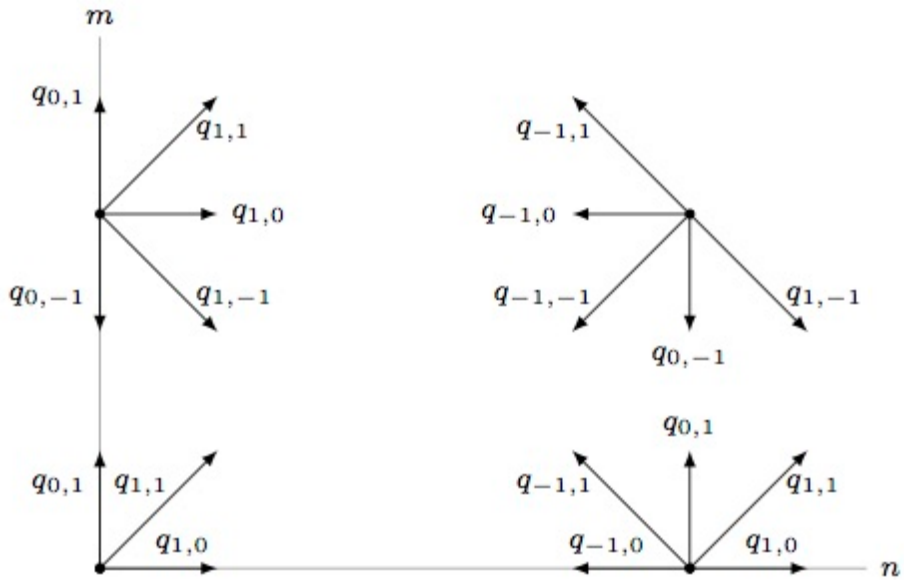
  [5] Smit, L.C. (2016). Steady State Analysis of Large-Scale Systems.

Main results

We consider the class of nearest neighbour random walks (NNRW)

- 1st quadrant
- Homogeneous nearest neighbour
- No transitions to N, NE and E

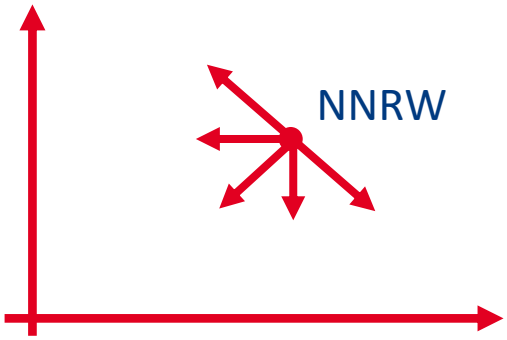




Main results

We consider the class of nearest neighbour random walks (NNRW)

- 1st quadrant
- Homogeneous nearest neighbour
- No transitions to N, NE and E



Theorem 1

We consider the class of NNRW and we calculate the eigenvalues and eigenvectors of $\boldsymbol{R}$ recursively.

Theorem 2

For the class of NNRW the infinite dimension rate matrix $\boldsymbol{R}$ is "diagonalizable" and we can numerically approximate $\boldsymbol{R}$ using spectral truncation.

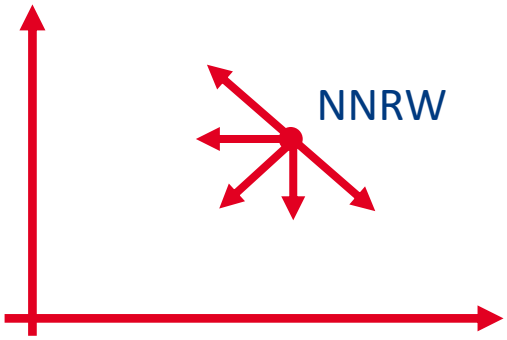
Theorem 3

We obtain the eigenvalues of the rate matrix for the original model.

Nearest neighbour random walk

We consider the class of nearest neighbour random walks (NNRW):

- 1st quadrant
- Homogeneous nearest neighbour
- No transitions to N, NE and E



Then,

$$\pi(m,n) \sim c\alpha^m \beta^n \text{ as } m,n \to \infty$$

More concretely,

$$\pi(m,n) = \sum_i c_i \alpha_i^m \beta_i^n, m,n > 0$$

The limitations above are necessary

[4] Adan, I.J.B.F. (1991). A Compensation Approach for Queueing Problems.

[6] Chen, Y. (2015). Random Walks in the Quarter-Plane: invariant Measures and Performance Bounds.

Boundary value method approach

First, introduce

$$\Pi(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \pi(m,n) x^m y^n$$

then

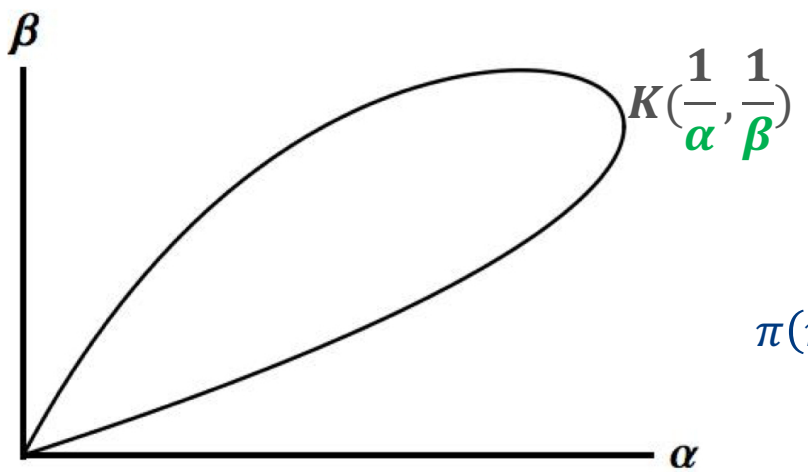
$$K(x,y)\Pi(x,y) = A(x,y)\Pi(x,0) + B(x,y)\Pi(0,y) + C(x,y)\Pi(0,0)$$

where $K(x,y), A(x,y), B(x,y), C(x,y)$ are known quadratic functions.

Choose $y = f(x)$, e.g. $y = \bar{x}$, and set $K(x, f(x)) = 0$

$$0 = A(x,f(x))\Pi(x,0) + B(x,f(x))\Pi(0,f(x)) + C(x,f(x))\Pi(0,0)$$

The above equation can be solved as a Riemann (Hilbert) boundary value problem.



$$\pi(m,n) = \sum_i c_i \alpha_i^m \beta_i^n, m,n > 0$$

Compensation approach

Aims at solving directly the balance equations of a random walk in the quadrant using a series (infinite or finite) of product-form solutions

Key idea:

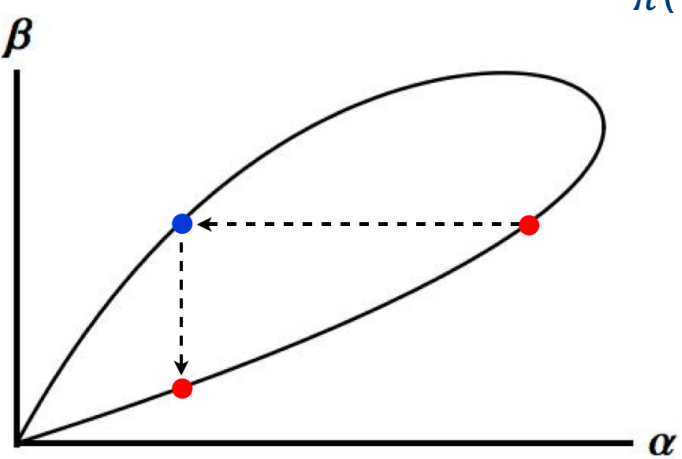
- Guess a product-form solution

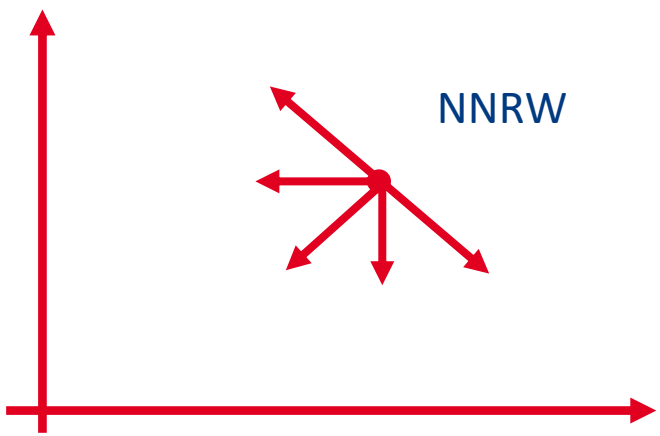
$$\alpha^m \beta^n$$

- Check if it satisfies the boundaries
- If not start compensating by adding new product-form terms

Solution

$$\pi(m,n) = \sum_i c_i \alpha_i^m \beta_i^n \, , m,n > 0$$





Matrix geometric approach

We know that

$$\boldsymbol{\pi}_m = \boldsymbol{\pi}_{m-1}\boldsymbol{R}$$

where $\boldsymbol{\pi}_m = (\pi(m,0)\ \ \pi(m,1)\ \ \dots)$ and $\pi(m,n) = \sum_i c_i \alpha_i^m \beta_i^n$, $m,n > 0$.

Then,

$$\Pi(x,y) = \boldsymbol{\pi}_0 \boldsymbol{y}' + \boldsymbol{\pi}_1 (x^{-1}\boldsymbol{I} - \boldsymbol{R})^{-1}\boldsymbol{y}'$$

where $\boldsymbol{y}' = (1\ y\ y^2\ \ \dots)$.

Substituting in the functional equation reveals

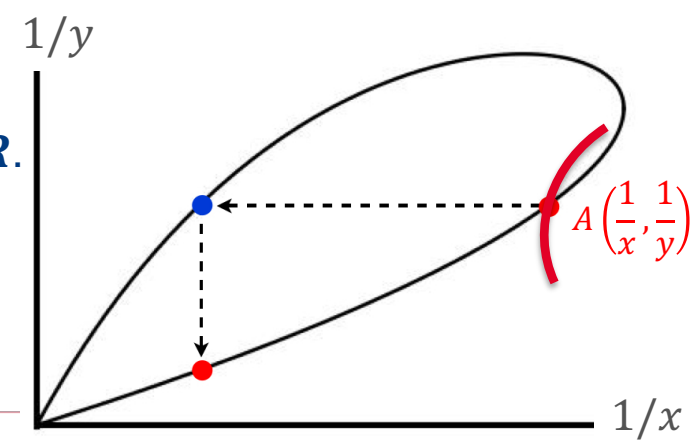
$$K(x,y)\Pi(x,y) = A(x,y)\Pi(x,0) + B(x,y)\Pi(0,y) + C(x,y)\Pi(0,0) \quad \Rightarrow$$

$$\boldsymbol{\pi}_1 (x^{-1}\boldsymbol{I} - \boldsymbol{R})^{-1}[K(x,y)\boldsymbol{y} + A(x,y)\boldsymbol{e}]$$
$$= -\boldsymbol{\pi}_0\big[\big(K(x,y) + B(x,y)\big)\boldsymbol{y}' + \big(A(x,y) + C(x,y)\big)\boldsymbol{e}'\big]$$

So $x^{-1} = \alpha$ is an eigenvalue of matrix $\boldsymbol{R}$.

The terms $y^{-1} = \beta$ *are associated with the eigenvectors of* $\boldsymbol{R}$.

$$\underbrace{\alpha_0 \longrightarrow \beta_0}_{\text{initial solution}} \underbrace{\longrightarrow \alpha_1}_{\text{vertical compensation}} \underbrace{\longrightarrow \beta_1}_{\text{horizontal compensation}} \quad \cdots$$



Matrix geometric approach

Theorem 1

The terms $\{\alpha_i\}$ constitute the different eigenvalues of the matrix $\boldsymbol{R}$. For eigenvalue α_i the corresponding eigenvector of the matrix $\boldsymbol{R}$ is $\boldsymbol{h_i}$ with $h_{i,n} = c_i(\beta_{i-1}^n + f_i\beta_i^n)$.

Theorem 2

Spectral decomposition

$$\boldsymbol{R} = \boldsymbol{H}^{-1}\boldsymbol{D}\boldsymbol{H}$$

Truncated spectral decomposition

$$\boldsymbol{R_I} = \boldsymbol{H_I}^{-1}\boldsymbol{D_I}\boldsymbol{H_I}$$

Remark

The latter is equivalent to truncating

$$\pi(m,n) = \sum_{i=0}^{I} c_i\alpha_i^m\beta_i^n\,, m,n > 0$$

Main results

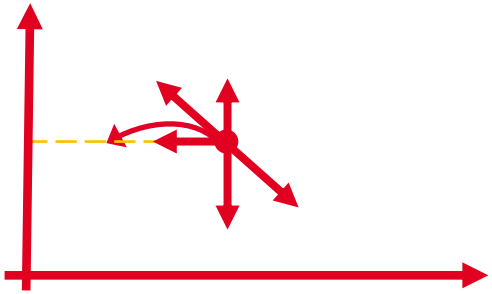
Theorem 3

We obtain the eigenvalues of the rate matrix for the original model.

$$K(x,y)\Pi(x,y) = A(x,y)\Pi(x,0) + B(x,y)\Pi(0,y) + C(x,y)\Pi(0,0) + D(x,y)\Pi(p+(1-p)x,y)$$

By using a similar argument as previously we obtain

$$\begin{tikzcd}
\boldsymbol{\alpha_0} \arrow[r] \arrow[d] & \boldsymbol{p+(1-p)\alpha_0} \arrow[r] \arrow[d] & \boldsymbol{p^2+(1-p^2)\alpha_0} \arrow[r] \arrow[d] & {} \\
\boldsymbol{\alpha_1} \arrow[r] \arrow[d] & \vdots \arrow[r] & \vdots & \\
{} & & &
\end{tikzcd}$$



Conclusions

- Calculation of eigenvalues and eigenvectors of rate matrix for NNRW
- Efficient numerical calculation of rate matrix using spectral truncation
- Our results show promise for "non-structured" $\boldsymbol{R}$ of random walks in the quadrant

Extensions

- Probabilistic interpretation of the product-form terms
- Use the results for approximation, i.e. approximate the invariant measure by a series (finite or infinite) of product forms.

[7] Y. Chen, R.J. Boucherie, and J. Goseling, (2016). Invariant measures and error bounds for random walks in the quarter-plane based on sums of geometric terms, arXiv:1502.07218.