4 Exhaustive-Service, Continuous-Time Systems

We now discuss the continuous-time version of the exhaustive service system. We consider the case of Poisson message arrivals (and only in Section 4.4.2 we find the mean message waiting time in a Poisson bulk arrival system).

Our model is as follows.

(i) An entity for service is a variable-length message. The length of each message at station i is assumed to be distributed according to a general distribution function whose LST is denoted by $B_i^*(s)$. Let b_i and $b_i^{(2)}$ be the mean and the second moment, respectively, for the message length at station i:

$$b_i = -B_i^{*(1)}(0), \qquad b_i^{(2)} = B_i^{*(2)}(0) \tag{4.1}$$

(ii) The message arrival process at station i is assumed to be Poisson with rate λ_i.

(iii) The reply interval between station i and station i+1 is assumed to be distributed according to a general distribution function whose LST is denoted by $R_i^*(s)$. Denote by r_i and δ_i^2 the mean and variance, respectively, for the reply interval between station i and station i+1:

$$r_i = -R_i^{*(1)}(0), \qquad \delta_i^2 = R_i^{*(2)}(0) - r_i^2. \tag{4.2}$$

In this chapter we consider the case of exhaustive service. Note that the gambler's ruin problem in the case of discrete-time systems (in Section 3.2) corresponds to the busy period analysis in an M/G/1 queue with arrival rate λ_i and service time distribution $B_i^*(s)$. Define $\Theta_i^*(s)$ as the LST of the distribution function for the busy period in such an M/G/1 queue. As shown in [Klei75,Sec.5.8] and [Coop81,Sec.5.8], it satisfies

$$\Theta_i^*(s) = B_i^*[s + \lambda_i - \lambda_i \Theta_i^*(s)] \tag{4.3a}$$

from which we have the first and second moments, θ_i and $\theta_i^{(2)}$, for the busy period as

$$\theta_i = -\Theta_i^{*(1)}(0) = \frac{b_i}{1-\rho_i}, \quad \theta_i^{(2)} = \Theta_i^{*(2)}(0) = \frac{b_i^{(2)}}{(1-\rho_i)^3} \tag{4.3b}$$

where

$$\rho_i \triangleq \lambda_i b_i \tag{4.4}$$

Also, define $\Gamma_i(z)$ as the GF for the number of messages served in a busy period in the same M/G/1 queue. From [Klei75,Sec.5.9], we have

$$\Gamma_i(z) = zB_i^*[\lambda_i - \lambda_i \Gamma_i(z)] \tag{4.5a}$$

from which

$$E[\Gamma_i] = \frac{1}{1-\rho_i}, \quad \text{Var}[\Gamma_i] = \frac{\rho_i(1-\rho_i) + \lambda_i^2 b_i^{(2)}}{(1-\rho_i)^3} \tag{4.5b}$$

4.1 Number of Messages at Polling Instants

Let

$$L_i(t) \triangleq \text{number of messages at station i at time t}$$

and define the joint and marginal GF's for $[L_1(t), L_2(t), \ldots, L_N(t)]$ at time $t=\tau_i(m)$, i.e., at the time when station i is polled as in (3.21a) and (3.21b), respectively.

In order to relate $F_i(z_1, z_2, \ldots, z_N)$ to $F_{i+1}(z_1, z_2, \ldots, z_N)$, first note that the service time for station i, $\bar{\tau}_i(m) - \tau_i(m)$, is distributed as the sum of $L_i(\tau_i(m))$ busy periods where each busy period is distributed as given in (4.3a). Thus we have

$$E[e^{-s\{\bar{\tau}_i(m)-\tau_i(m)\}}] = [\Theta_i^*(s)]^{L_i(\tau_i(m))} \tag{~~4.6a~~}$$

Therefore the joint GF (except for station i) for the number of arrivals during the service time for station i is given by

$$\left\{\Theta_i^*\left[\sum_{\substack{j=1\\(j\neq i)}}^{N} (\lambda_j - \lambda_j z_j)\right]\right\}^{L_i(\tau_i(m))} \tag{4.6b}$$

Similarly, the joint GF for the number of message arrivals during the reply interval between station i and station i+1 is given by

$$R_i^*\left[\sum_{j=1}^{N} (\lambda_j - \lambda_j z_j)\right] \tag{4.7}$$

Hence, by an argument similar to Section 3.3, we get [Hash72a]

$$F_{i+1}(z_1,z_2,\ldots,z_N)=R_i^*\left[\sum_{j=1}^{N} (\lambda_j-\lambda_j z_j)\right]$$

$$\cdot\ F_i\left(z_1,z_2,\ldots,z_{i-1},\Theta_i^*\left[\sum_{\substack{j=1\\(j\neq i)}}^{N} (\lambda_j-\lambda_j z_j)\right], z_{i+1},\ldots,z_N\right) \tag{4.8}$$

Now, define the moments for $[L_1(t),L_2(t),\ldots,L_N(t)]$ at $t=\tau_i(m)$ as in (3.27), (3.28a) and (3.28b). From (4.8), a set of N^2 equations for $\{f_i(j);\ i,j = 1,2,\ldots,N\}$ is given by

$$f_{i+1}(i) = r_i\lambda_i \tag{4.9a}$$

$$f_{i+1}(j) = r_i\lambda_j + f_i(j) + \frac{f_i(i)\lambda_j b_i}{1-\rho_i} \qquad j\neq i \tag{4.9b}$$

(Note that (4.9a) and (4.9b) have the same form as (3.29a) and (3.29b).)

The solution to (4.9a) and (4.9b) is given by

$$E[L_i^*] = f_i(i) = \frac{\lambda_i(1-\rho_i)\sum_{k=1}^{N} r_k}{1-\sum_{k=1}^{N}\rho_k} \tag{4.10a}$$

$$f_i(j) = \lambda_j \left[\sum_{k=j}^{i-1} r_k + \frac{[\sum_{k=j+1}^{i-1}\rho_k][\sum_{k=1}^{N} r_k]}{1-\sum_{k=1}^{N}\rho_k} \right] \qquad j\neq i \tag{4.10b}$$

A set of N^3 equations for $\{f_i(j,k);\ i,j,k = 1,2,\ldots,N\}$ is given by

$$f_{i+1}(j,k) = \lambda_j\lambda_k(\delta_i^2+r_i^2) + r_i\lambda_k f_i(j) + r_i\lambda_j f_i(k) + f_i(i)\lambda_j\lambda_k\left[\frac{2r_ib_i}{1-\rho_i} + \frac{b_i^{(2)}}{(1-\rho_i)^3}\right]$$
$$+ \frac{b_i}{1-\rho_i}[f_i(i,j)\lambda_k + f_i(i,k)\lambda_j] + f_i(j,k) + \frac{f_i(i,i)\lambda_j\lambda_k b_i^2}{(1-\rho_i)^2} \qquad i\neq j,\ j\neq k \tag{4.11a}$$

$$f_{i+1}(i,k) = \lambda_i\lambda_k(\delta_i^2+r_i^2) + r_i\lambda_i\left[f_i(k) + \frac{f_i(i)\lambda_k b_i}{1-\rho_i}\right] \qquad i\neq k \tag{4.11b}$$

$$f_{i+1}(i,i) = \lambda_i^2(\delta_i^2 + r_i^2) \tag{4.11c}$$

In the case of identical stations, we have [Hash72a]

$$E[L^*] = \frac{Nr\lambda(1-\rho)}{1-N\rho} \tag{4.12a}$$

$$f_i(i,i) = \frac{\delta^2\lambda^2N(1-\rho)}{1-N\rho} + \frac{N(N-1)\lambda^3 rb^{(2)}}{(1-N\rho)^2} + \frac{N^2r^2\lambda^2(1-\rho)^2}{(1-N\rho)^2} \tag{4.12b}$$

4.2 Service Time, Intervisit Time and Cycle Time

The LST of the distribution function for the service time at station i,

S_i, is given by (4.6a); i.e.,

$$S_i^*(s) = F_i[\Theta_i^*(s)] \qquad (4.13)$$

from which we have

$$E[S_i] = E[L_i^*]\theta_i = \frac{\rho_i \sum_{k=1}^{N} r_k}{1 - \sum_{k=1}^{N} \rho_k} \qquad (4.14a)$$

$$Var[S_i] = \theta_i^2 \, Var[L_i^*] + (\theta_i^{(2)} - \theta_i^2)\, E[L_i^*]$$

$$= \frac{1}{(1-\rho_i)^2}\left[b_i^2 \, Var[L_i^*] + \frac{\lambda_i[b_i^{(2)} - b_i^2(1-\rho_i)] \sum_{k=1}^{N} r_k}{1 - \sum_{k=1}^{N} \rho_k} \right] \qquad (4.14b)$$

where we have used (4.3b) and (4.10a). In the case of identical stations, by using (4.12b) we have

$$E[S] = \frac{Nr\rho}{1-N\rho} \qquad (4.15a)$$

$$Var[S] = \frac{\delta^2 N\rho^2}{(1-\rho)(1-N\rho)} + \frac{N\lambda r b^{(2)}(1+\rho-N\rho)}{(1-\rho)(1-N\rho)^2} \qquad (4.15b)$$

Note that (4.15a) and (4.15b) correspond to (3.34a) and (3.34b), respectively, in the discrete-time system.

We next give the GF for the number of messages served in a service time for station i, denoted by $T_i(z)$. Since these messages are those served in a busy period initiated by $L_i(\tau_i(m))$ messages, it follows that

$$T_i(z) = F_i[\Gamma_i(z)] \qquad (4.16)$$

where $\Gamma_i(z)$ is given by (4.5a). Using (4.5b) and (4.10a), we have

$$E[T_i(m)] = E[\Gamma_i]E[L_i^*] = \frac{\lambda_i \sum_{k=1}^{N} r_k}{1-\sum_{k=1}^{N} \rho_k} \tag{4.17a}$$

$$Var[T_i(m)] = Var[L_i^*](E[\Gamma_i])^2 + E[L_i^*]\ Var\ [\Gamma_i]$$

$$= \frac{1}{(1-\rho_i)^2}\left[Var[L_i^*] + \frac{\lambda_i[\rho_i(1-\rho_i)+\lambda_i^2 b_i^{(2)}] \sum_{k=1}^{N} r_k}{1-\sum_{k=1}^{N}\rho_k}\right] \tag{4.17b}$$

In the case of identical stations, by using (4.12b), we get

$$E[T(m)] = \frac{N\lambda r}{1-N\rho} \tag{4.18a}$$

$$Var[T(m)] = \frac{\delta^2 N\lambda^2}{(1-\rho)(1-N\rho)} + \frac{N^2 r\lambda^3 b^{(2)}}{(1-\rho)(1-N\rho)^2} + \frac{N\lambda r(1+\rho)}{(1-\rho)(1-N\rho)} \tag{4.18b}$$

We proceed to consider $I_i^*(s)$, the LST for distribution function of the intervisit time for station i. Since the number of message arrivals during the intervisit time is the number of messages found at the polling instant, we have the relation

$$I_i^*(\lambda_i - \lambda_i z) = F_i(z) \tag{4.19}$$

or, by $s=\lambda_i - \lambda_i z$,

$$I_i^*(s) = F_i(1 - s/\lambda_i) \tag{4.20a}$$

from which we have

$$E[I_i] = \frac{f_i(i)}{\lambda_i} = \frac{(1-\rho_i)\sum_{k=1}^{N} r_k}{1-\sum_{k=1}^{N}\rho_k}, \quad E[\{I_i\}^2] = \frac{f_i(i,i)}{\lambda_i^2} \tag{4.20b}$$

In the case of identical stations, by using (4.12b), we have

$$E[I] = \frac{Nr(1-\rho)}{1-N\rho}, \quad Var[I] = \frac{\delta^2 N(1-\rho)}{1-N\rho} + \frac{N(N-1)\lambda r b^{(2)}}{(1-N\rho)^2} \tag{4.21}$$

which correspond to (3.37b) in the discrete-time system.

The LST of the distribution function for the cycle time for station i, C_i, may be obtained in the same way as (3.39b) is derived. If we denote by $I_i(t)$ the distribution function for I_i, then

$$C_i^*(s) \triangleq E[e^{-s\{\bar{\tau}_i(m+1) - \bar{\tau}_i(m)\}}]$$

$$= E[e^{-s\{\bar{\tau}_i(m+1)-\tau_i(m+1)\}} e^{-s\{\tau_i(m+1)-\bar{\tau}_i(m)\}}]$$

$$= \int_0^\infty E[e^{-s\{\bar{\tau}_i(m+1)-\tau_i(m+1)\}} e^{-st} \,|\, \tau_i(m+1) - \bar{\tau}_i(m)=t]\, dI(t)$$

However, by applying (4.13) and (4.19), we have

$$E[e^{-s\{\bar{\tau}_i(m+1)-\tau_i(m+1)\}} \mid \tau_i(m+1) - \bar{\tau}_i(m)=t]$$

$$= E[\{\Theta_i^*(s)\}^{\tau_i(m+1)} \mid \tau_i(m+1)-\bar{\tau}_i(m)=t]$$

$$= E[e^{-\{\lambda_i-\lambda_i\Theta_i^*(s)\}\{\tau_i(m+1)-\bar{\tau}_i(m)\}} \mid \tau_i(m+1)-\bar{\tau}_i(m)=t]$$

$$= e^{-\{\lambda_i-\lambda_i\Theta_i^*(s)\}t}$$

It follows that [Hash81b]

$$C_i^*(s) = I_i^*[s + \lambda_i - \lambda_i\Theta_i^*(s)] \tag{4.22}$$

which corresponds to (3.39b) in the discrete time system. From (4.22), we have

$$E[C_i] = (1+\lambda_i\theta_i)E[I_i] = \frac{\sum_{k=1}^{N} r_k}{1-\sum_{k=1}^{N} \rho_k} \tag{4.23a}$$

$$\mathrm{Var}\,[C_i] = (1+\lambda_i\theta_i)^2\ \mathrm{Var}\,[I_i] + \lambda_i\theta_i^{(2)}\ E[I_i]$$

$$= \frac{1}{\lambda_i^2(1-\rho_i)^2}\left[\mathrm{Var}[L_i^*] + \frac{[-(1-\rho_i)+\lambda_i^2 b_i^{(2)}]\sum_{k=1}^{N} r_k}{1-\sum_{k=1}^{N}\rho_k}\right] \tag{4.23b}$$

where we have used (4.3b) and (4.20b). In the case of identical stations, by using (4.12b), we obtain

$$E[C] = \frac{Nr}{1-N\rho},\quad \mathrm{Var}[C] = \frac{\delta^2 N}{(1-\rho)(1-N\rho)} + \frac{N^2\lambda r b^{(2)}}{(1-\rho)(1-N\rho)^2} \tag{4.24}$$

which corresponds to (3.41) in the discrete time system. From (4.23a), the stability condition is given by

$$\sum_{k=1}^{N} \rho_k < 1 \tag{4.25}$$

4.3 Number of Messages at Departure Instants

Let us define a sequence of times $\{\tau_i^{(n)}(m); n=1,2,\ldots,T_i(m)\}$, where $\tau_i^{(n)}(m)$ is the service completion instant for the n th message served in the m th cycle at station i. $T_i(m)$ is the number of messages served in the m th cycle at station i. We now consider the GF, $Q_i(z)$, for the number of messages just after the moment of message service completion at station i, which is denoted by L_i. It is given by the average of $z^{L_i(\tau_i^{(n)}(m))}$ over the average number of messages served in the m th cycle, where the averages are taken over the regenerative cycle $\{C_i^{(m)}, m=1,2,\ldots,M_i\}$. The regeneration points are those when station i is polled and all $L_j(\tau_i(m))$, $1\le j\le N$, are zero (see Section 3.4.1 for a similar discussion in the discrete-time system). Thus we have

$$Q_i(z) \triangleq E[z^{L_i}] = \frac{E\left\{\sum_{m=1}^{M_i}\sum_{n=1}^{T_i(m)} z^{L_i(\tau_i^{(n)}(m))}\right\}}{E\left\{\sum_{m=1}^{M_i} T_i(m)\right\}} \tag{4.26}$$

However, since M_i is a stopping time for the regenerative process with $E[M_i] < \infty$, it follows from Wald's lemma that

$$E\left[\sum_{m=1}^{M_i}\sum_{n=1}^{T_i(m)} z^{L_i(\tau_i^{(n)}(m))}\right] = E[M_i]\, E\left[\sum_{n=1}^{T_i(m)} z^{L_i(\tau_i^{(n)}(m))}\right] \tag{4.27a}$$

$$E\left[\sum_{m=1}^{M_i} T_i(m)\right] = [M_i]\, E[T_i(m)] \tag{4.27b}$$

Thus we get

$$Q_i(z) \triangleq E[z^{L_i}] = \frac{E\left\{\sum_{n=1}^{T_i(m)} z^{L_i(\tau_i^{(n)}(m))}\right\}}{E[T_i(m)]} \tag{4.28}$$

The denominator in (4.28) is given in (4.17a).

To evaluate the sum in the numerator of (4.28), we apply the formula (3.20b) with substitution

$$T \to T_i(m), \quad L_n \to L_i(\tau_i^{(n)}(m)), \quad P(z) \to B_i^*(\lambda_i - \lambda_i z)$$

$$F(z) = E[z^{Lo}] \to F_i(z) = E[z^{L_i(\tau_i(m))}]$$

Then we get

$$E\left[\sum_{n=1}^{T_i(m)} z^{L_i(\tau_i^{(n)}(m))}\right] = \frac{B_i^*(\lambda_i-\lambda_i z)}{z-B_i^*(\lambda_i-\lambda_i z)}\,[F_i(z)-1]$$

Substituting this expression and (4.17a) into (4.28), we have

$$Q_i(z) = \frac{1-\sum_{k=1}^{N}\rho_k}{\lambda_i \sum_{k=1}^{N} r_k} \cdot \frac{B_i^*(\lambda_i-\lambda_i z)}{z-B_i^*(\lambda_i-\lambda_i z)}\,[F_i(z)-1] \qquad (4.29)$$

From (4.29) we easily get the average number of messages at station i left behind the service completion at station i:

$$E[L_i] = \rho_i + \frac{\lambda_i^2 b_i^{(2)}}{2(1-\rho_i)} + \frac{[1-\sum_{k=1}^{N}\rho_k] f_i(i,i)}{2\lambda_i(1-\rho_i)\sum_{k=1}^{N} r_k} \qquad (4.30a)$$

In the case of identical stations, we have

$$E[L] = \rho + \frac{\lambda\delta^2}{2r} + \frac{N\lambda r(1-\rho)}{2(1-N\rho)} + \frac{N\lambda^2 b^{(2)}}{2(1-N\rho)} \qquad (4.30b)$$

Although (4.29), (4.30a) and (4.30b) have been obtained for the number of messages left by departing messages, we claim that they hold for the number of messages at an arbitrary instant. This claim comes from the assumption of Poisson arrivals and from the fact that the state changes by unit step values (one by one) only. In such a case, as shown in [Klei75, Sec.5.3 and Prob.5.6], the arrival-time, arbitrary-time and departure-time probabilities have the same limiting distribution.

4.4 Waiting Time

Finally we may derive $W_i^*(s)$, the LST of the distribution function for the waiting time W_i (excluding the service time) of each message at station i. Note that the number of message arrivals at station i during the period in which a message stays at station i (the LST of the distribution function for this period is given by $W_i^*(s)B_i^*(s)$) is equal to the number of messages left behind at the service completion time for that message. Hence, we have

$$W_i^*(\lambda_i - \lambda_i z)B_i^*(\lambda_i - \lambda_i z) = Q_i(z) \tag{4.31}$$

Letting $s=\lambda_i - \lambda_i z$, we get

$$W_i^*(s) = \frac{Q_i(1-s/\lambda_i)}{B_i^*(s)} = \frac{1-\sum_{k=1}^{N}\rho_k}{\sum_{k=1}^{N} r_k} \cdot \frac{1-F_i(1-s/\lambda_i)}{s-\lambda_i+\lambda_i B_i^*(s)} \tag{4.32}$$

From (4.32) we have

$$E[W_i] = \frac{\lambda_i b_i^{(2)}}{2(1-\rho_i)} + \frac{[1-\sum_{k=1}^{N}\rho_k]f_i(i,i)}{2\lambda_i^2(1-\rho_i)\sum_{k=1}^{N} r_k} \tag{4.33a}$$

Comparing (4.30a) and (4.33a), we may confirm Little's result [Litt61]

$$E[L_i] = \lambda_i\, E[W_i] + \rho_i \tag{4.34}$$

We note that Little's result applies to the mean number of messages at an arbitrary time.