5.2 Continuous-Time Systems (gated)

This section deals with continuous-time systems; the assumptions and notation for the message length, message arrival process and reply interval are carried over from Chapter 4.

We first find the relation between $F_i(z_1, z_2, \ldots, z_N)$ and $F_{i+1}(z_1, z_2, \ldots, z_N)$. In the case of a gated service system, the number of messages served in the service time for station i, $\bar{\tau}_i(m) - \tau_i(m)$, is simply $L_i(\tau_i(m))$, and each message service takes a time whose pdf has a transform given by $B_i^*(s)$. Thus, we have

$$E[e^{-s\{\bar{\tau}_i(m)-\tau_i(m)\}}] = [B_i^*(s)]^{L_i(\tau_i(m))} \qquad (5.27a)$$

Hence the joint GF for the number of message arrivals during the service period for station i is given by

$$\{B_i^*[\sum_{j=1}^{N} (\lambda_j - \lambda_j z_j)]\}^{L_i(\tau_i(m))} \qquad (5.27b)$$

The joint GF for the number of message arrivals during the reply interval between station i and station i+1 is given by

$$R_i^*[\sum_{j=1}^{N} (\lambda_j - \lambda_j z_j)] \qquad (5.28)$$

Hence, we have [Hash70]

$$F_{i+1}(z_1,z_2,\ldots,z_N)=R_i^*[\sum_{j=1}^{N}(\lambda_j-\lambda_j z_j)]\cdot$$
$$\cdot F_i(z_1,z_2,\ldots,z_{i-1},B_i^*[\sum_{j=1}^{N}(\lambda_j-\lambda_j z_j)], z_{i+1},\ldots,z_N) \qquad (5.29)$$

Now, define the moments for $[L_1(t), L_2(t), \ldots, L_N(t)]$ at time $t=\tau_i(m)$ as in (3.27), (3.28a) and (3.28b). A set of N^2 equations for $\{f_i(j);\ i,j=$

$1,2,\ldots,N\}$ is given by

$$f_{i+1}(i) = r_i\lambda_i + \rho_i f_i(i) \tag{5.30a}$$

$$f_{i+1}(j) = r_i\lambda_j + \lambda_j b_i f_i(i) + f_i(j) \qquad j\neq i \tag{5.30b}$$

The solution to (5.30a) and (5.30b) is

$$E[L_i^*] = f_i(i) = \frac{\lambda_i \sum_{k=1}^{N} r_k}{1 - \sum_{k=1}^{N} \rho_k} \tag{5.31a}$$

$$f_i(j) = \lambda_j \left[\sum_{k=j}^{i-1} r_k + \frac{(\sum_{k=j}^{i-1}\rho_k)(\sum_{k=1}^{N} r_k)}{1 - \sum_{k=1}^{N} \rho_k} \right] \qquad j\neq i \tag{5.31b}$$

A set of N^3 equations for $\{f_i(j,k); i,j,k=1,2,\ldots,N\}$ is given by

$$f_{i+1}(j,k)=\lambda_j\lambda_k(\delta_i^2+r_i^2)+r_i\lambda_k f_i(j)+r_i\lambda_j f_i(k)+f_i(i)\lambda_j\lambda_k(2r_ib_i+b_i^{(2)})+f_i(j,k)$$
$$+b_i\lambda_k f_i(i,j)+b_i\lambda_j f_i(i,k)+b_i^2\lambda_j\lambda_k f_i(i,i) \qquad i\neq j,\ i\neq k \tag{5.32a}$$

$$f_{i+1}(i,k)=\lambda_i\lambda_k(\delta_i^2+r_i^2)+r_i\lambda_i f_i(k)+f_i(i)\lambda_i\lambda_k(2r_ib_i+b_i^{(2)})+\lambda_i b_i f_i(i,k)$$
$$+\lambda_i\lambda_k b_i^2 f_i(i,i) \qquad i\neq k \tag{5.32b}$$

$$f_{i+1}(i,i)=\lambda_i^2(\delta_i^2+r_i^2)+f_i(i)\lambda_i^2(2r_ib_i+b_i^{(2)})+(\lambda_i b_i)^2 f_i(i,i) \tag{5.32c}$$

In the case of identical stations, we have [Hash70]

$$E[L^*] = \frac{Nr\lambda}{1 - N\rho} \tag{5.33a}$$

$$f_i(i,i) = \frac{\delta^2\lambda^2 N}{(1-N\rho)(1+\rho)} + \frac{N^2\lambda^3 r b^{(2)}}{(1-N\rho)^2(1+\rho)} + \frac{N^2 r^2\lambda^2}{(1-N\rho)^2} \tag{5.33b}$$

Let us now consider the service time, intervisit time, cycle time, and number of messages served in a cycle. First, the LST of the distribution function for the service time for station i is given by (5.27a), i.e.,

$$S_i^*(s) = F_i[B_i^*(s)] \tag{5.34}$$

from which we have

$$E[S_i] = b_i E[L_i^*] = \frac{\rho_i \sum_{k=1}^{N} r_k}{1 - \sum_{k=1}^{N} \rho_k} \tag{5.35a}$$

$$\mathrm{Var}[S_i] = b_i^2 \mathrm{Var}[L_i^*] + (b_i^{(2)} - b_i^2) E[L_i^*] \tag{5.35b}$$

In the case of identical stations, by using (5.33b), we have

$$E[S] = \frac{Nr\rho}{1-N\rho}, \quad \mathrm{Var}[S] = \frac{\delta^2 N\rho^2}{(1+\rho)(1-N\rho)} + \frac{N\lambda r b^{(2)}(1+\rho-N\rho)}{(1+\rho)(1-N\rho)^2} \tag{5.36}$$

which corresponds to (5.6) in the discrete-time system.

The number of messages served in a cycle is exactly the number of messages found at the polling instant:

$$T_i(m) = L_i(\tau_i(m)) \tag{5.37a}$$

and so, by (5.31a), we have

$$E[T_i(m)] = E[L_i(\tau_i(m))] = E[L_i^*] = \frac{\lambda_i \sum_{k=1}^{N} r_k}{1 - \sum_{k=1}^{N} \rho_k} \tag{5.37b}$$

The LST of the distribution function for the cycle time for station i is given through a relationship similar to (5.7):

$$F_i(z) = C_i^*(\lambda_i - \lambda_i z) \tag{5.38}$$

or

$$C_i^*(s) = F_i(1 - s/\lambda_i) \tag{5.39a}$$

from which we have

$$E[C_i] = \frac{E[L_i^*]}{\lambda_i} = \frac{\sum_{k=1}^{N} r_k}{1 - \sum_{k=1}^{N} \rho_k} \quad , \quad E[\{C_i\}^2] = \frac{f_i(i,i)}{\lambda_i^2} \tag{5.39b}$$

In the case of identical stations, by use of (5.33b), we have

$$E[C] = \frac{Nr}{1-N\rho} \quad , \quad Var[C] = \frac{\delta^2 N}{(1-N\rho)(1+\rho)} + \frac{N^2 \lambda r b^{(2)}}{(1-N\rho)^2 (1+\rho)} \tag{5.40}$$

which corresponds to (5.9) in the discrete-time system, and to (4.24) in the exhaustive service system. From (5.39b), the stability condition is given by (4.25).

As for the intervisit time, we only know its mean:

$$E[I_i] = E[C_i] - E[S_i] = \frac{(1-\rho_i) \sum_{k=1}^{N} r_k}{1 - \sum_{k=1}^{N} \rho_k} = (1-\rho_i)E[C_i] \tag{5.41}$$

Compare (4.14a) and (5.35a); (4.20b) and (5.41); (4.23a) and (5.39b); (4.17a) and (5.37b). We see that they are respectively identical.

We next derive the GF, $Q_i(z)$, for the number of messages left at station i after the message service completion at station i. By the same regenerative arguments as in Section 4.3, $Q_i(z)$ is given by (4.28). Note that the denominator is given by (5.37b). To evaluate the sum in the numerator of (4.28), note that

$$L_i(\tau_i^{(n)}(m)) = L_i(\tau_i(m)) - n + \text{number of arrivals during n service times} \tag{5.42a}$$

Using (5.37a), we have

$$E\left[\sum_{n=1}^{T_i(m)} z^{L_i(\tau_i^{(n)}(m))}\right] = E\left[\sum_{n=1}^{T_i(m)} z^{T_i(m)-n}\{B_i^*(\lambda_i-\lambda_i z)\}^n\right]$$

$$= \frac{B_i^*(\lambda_i-\lambda_i z)}{z-B_i^*(\lambda_i-\lambda_i z)}\, E\left[z^{T_i(m)} - \{B_i^*(\lambda_i-\lambda_i z)\}^{T_i(m)}\right]$$

$$= \frac{B_i^*(\lambda_i-\lambda_i z)}{z-B_i^*(\lambda_i-\lambda_i z)}\{F_i(z) - F_i[B_i^*(\lambda_i-\lambda_i z)]\} \tag{5.42b}$$

Using (5.37b) and (5.42b) in (4.28), we get

$$Q_i(z) = \frac{1-\sum_{k=1}^{N}\rho_k}{\lambda_i \sum_{k=1}^{N} r_k} \cdot \frac{B_i^*(\lambda_i - \lambda_i z)}{z - B_i^*(\lambda_i - \lambda_i z)} \cdot \{F_i(z)-F_i[B_i^*(\lambda_i - \lambda_i z)]\} \tag{5.43}$$

From (5.43) we obtain

$$E[L_i] = \rho_i + \frac{(1-\sum_{k=1}^{N}\rho_k)(1+\rho_i)\, f_i(i,i)}{2\lambda_i \sum_{k=1}^{N} r_k} \tag{5.44a}$$

In the case of identical stations, by using (5.33b) we have

$$E[L] = \rho + \frac{\delta^2\lambda}{2r} + \frac{Nr\lambda(1+\rho)}{2(1-N\rho)} + \frac{N\lambda^2 b^{(2)}}{2(1-N\rho)} \tag{5.44b}$$

We have derived (5.43), (5.44a) and (5.44b) for the number of messages at message departure instants. However, for the same reason mentioned after (4.30b), they hold at instant chosen at random.

By an argument similar to Section 4.4, we may find $W_i^*(s)$, the LST of the distribution function for the message waiting time at station i. Using (5.43), we have

$$W_i^*(s) = \frac{Q_i(1-s/\lambda_i)}{B_i^*(s)} = \frac{1-\sum_{k=1}^{N}\rho_k}{\sum_{k=1}^{N} r_k} \cdot \frac{F_i[B_i^*(s)]-F_i(1-s/\lambda_i)}{s-\lambda_i+\lambda_i B_i^*(s)} \tag{5.45}$$

From (5.45), we get

$$E[W_i] = \frac{(1-\sum_{k=1}^{N}\rho_k)(1+\rho_i)f_i(i,i)}{2\lambda_i^2 \sum_{k=1}^{N} r_k} \tag{5.46a}$$

From (5.44a) and (5.46a) we may again confirm Little's result (4.34).

In the case of identical stations, by using (5.33b) we have [Hash70],[Hash72a]

$$E[W] = \frac{\delta^2}{2r} + \frac{Nr(1+\rho)}{2(1-N\rho)} + \frac{N\lambda b^{(2)}}{2(1-N\rho)} \tag{5.46b}$$

Again, note that (5.46b) has the same form as (5.23) when Poisson arrivals are assumed ($\gamma^2=\lambda$) and "a half slot" (i.e., the term $\frac{1}{2}$ in (5.23)) is made shrink to zero in (5.23).