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Matrix geometric approach for random walks in the quadrant

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Joint work with
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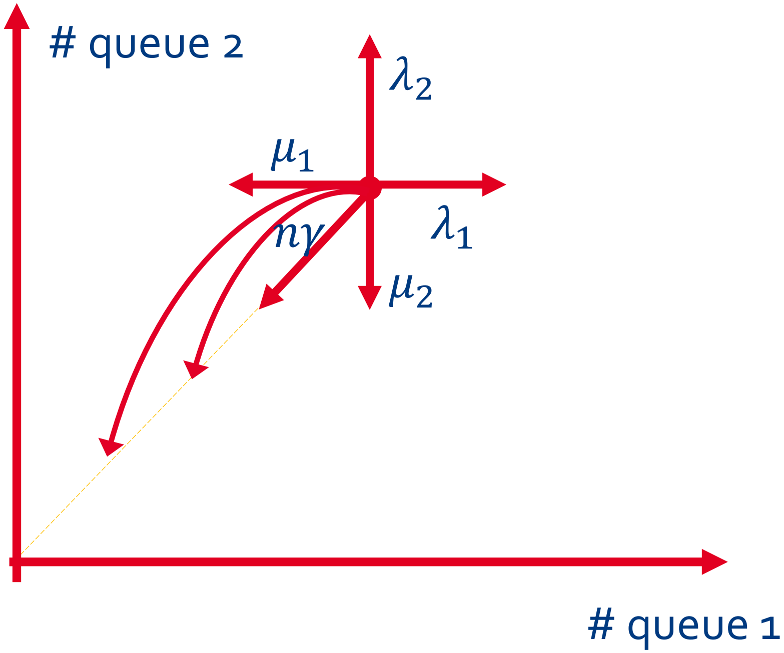
Where innovation starts

Motivation

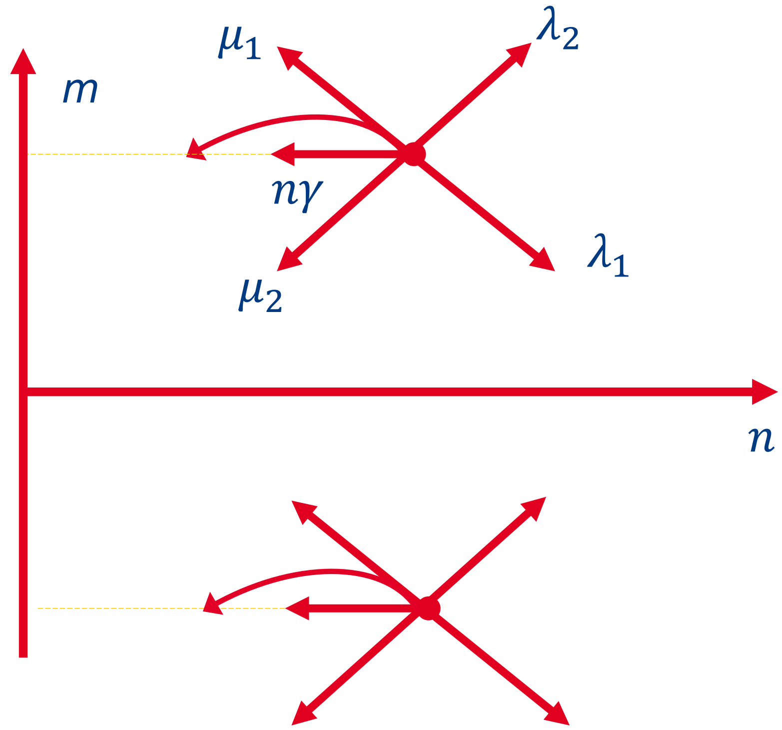
Performance analysis of

two coupled M / M /1 queues (in parallel), where the coupling occurs due to simultaneous abandonments



We transform the state space description

$n = \min\{q_1, q_2\}$ and $m = q_1 - q_2$



Objective: determine the steady-state distribution $\pi(n, m)$

Background

Exact analysis techniques for random walks

- Boundary value method approach

  [1] Cohen, J.W. and Boxma, O.J. (1983). Boundary Value Problems in Queueing System Analysis.

  [2] Fayolle, G., Iasnogorodski, R. and Malyshev, V. (1999). Random Walks in the Quarter Plane.

- Matrix geometric approach

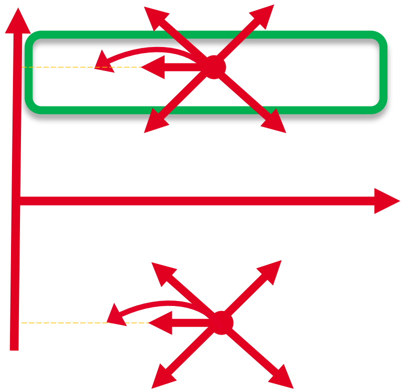
$$A_1 + RA_0 + R^2A_{-1} = 0$$

- Compensation approach

  [3] Adan, I.J.B.F. (1991). A Compensation Approach for Queueing Problems.

- Successive lumping

  [4] Smit, L.C. (2016). Steady State Analysis of Large-Scale Systems.

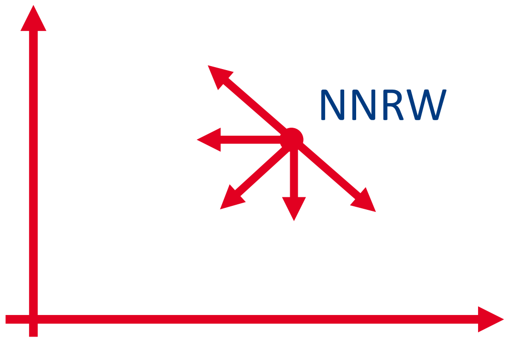


The above techniques have been developed separately and although there exists a set of models for which all aforementioned techniques are appropriate they haven't been connected!

Main results

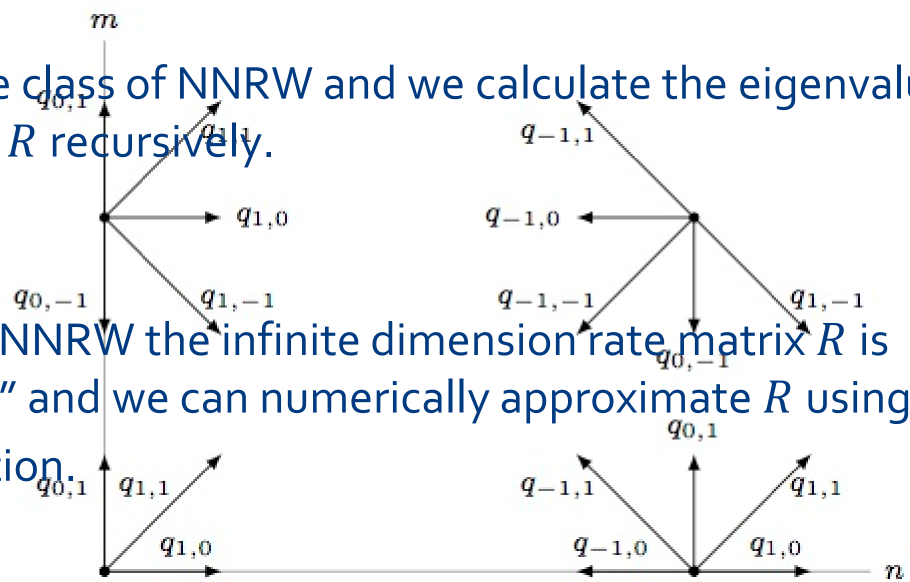
We consider the class of nearest neighbour random walks (NNRW) and we connect

- Boundary value method approach
- Matrix geometric approach
- Compensation approach



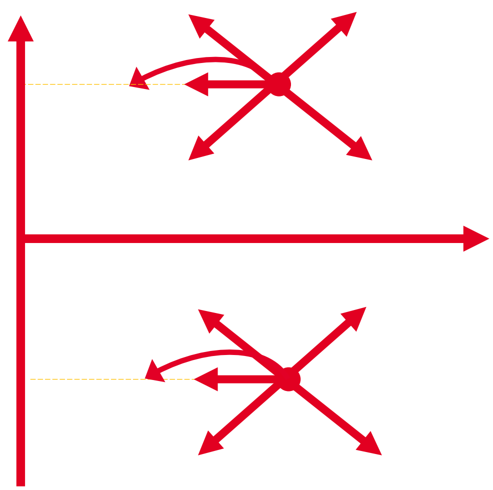
Theorem 1

We consider the class of NNRW and we calculate the eigenvalues and eigenvectors of R recursively.



Theorem 2

For the class of NNRW the infinite dimension rate matrix R is "diagonalizable" and we can numerically approximate R using spectral truncation.



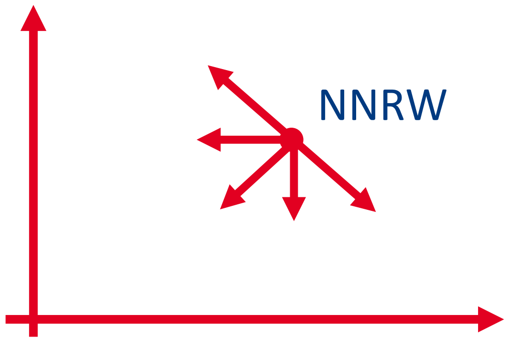
Theorem 3

We obtain the eigenvalues of the rate matrix for the original model.

Nearest neighbour random walk

We consider the class of nearest neighbour random walks (NNRW):

- 1st quadrant
- Homogeneous nearest neighbour
- No transitions to N, NE and E



Then,

$$\pi(n,m) \sim c\alpha^n\beta^m \text{ as } n,m \to \infty$$

More concretely,

$$\pi(n,m) = \sum_i c_i \alpha_i^n \beta_i^m, n,m > 0$$

The limitations above are sufficient

[3] Adan, I.J.B.F. (1991). A Compensation Approach for Queueing Problems.

and necessary

[5] Chen, Y. (2015). Random Walks in the Quarter-Plane: invariant Measures and Performance Bounds.

Boundary value method approach

First, introduce

$$\Pi(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\pi(n,m)x^n y^m$$

then

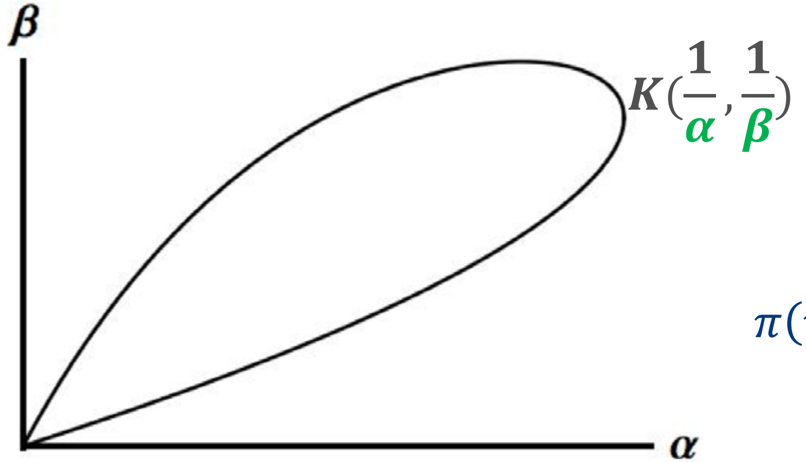
$$K(x,y)\Pi(x,y) = A(x,y)\Pi(x,0) + B(x,y)\Pi(0,y) + C(x,y)\Pi(0,0)$$

where $K(x,y), A(x,y), B(x,y), C(x,y)$ are known quadratic functions.

Choose $y = f(x)$, e.g. $y = \bar{x}$, and set $K(x, f(x)) = 0$

$$0 = A(x,f(x))\Pi(x,0) + B(x,f(x))\Pi(0,f(x)) + C(x,f(x))\Pi(0,0)$$

The above equation can be solved as a Riemann (Hilbert) boundary value problem.



$$\pi(n,m) = \sum_i c_i \alpha_i^n \beta_i^m, n,m > 0$$

Compensation approach

Aims at solving directly the balance equations of a random walk in the quadrant using a series (infinite or finite) of product-form solutions

Key idea:

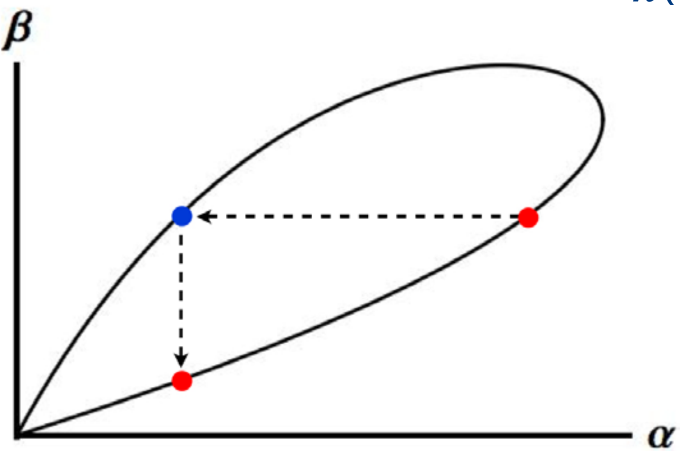
- Guess a product-form solution

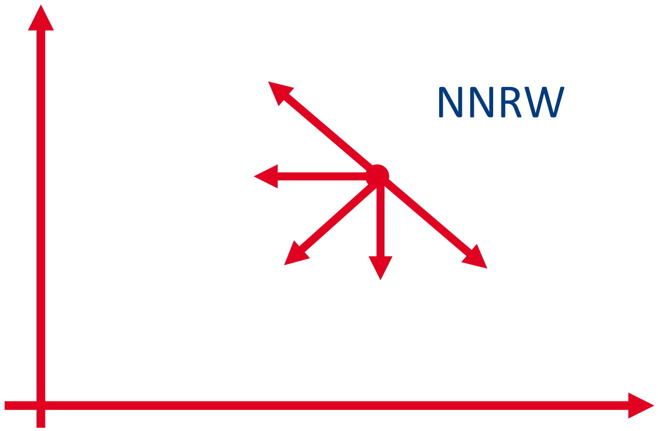
$$\alpha^n \beta^m$$

- Check if it satisfies the boundaries
- If not start compensating by adding new product-form terms

Solution

$$\pi(n,m) = \sum_i c_i \alpha_i^n \beta_i^m \,, n, m > 0$$





Matrix geometric approach

We know that

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_{n-1}\boldsymbol{R}$$

where $\boldsymbol{\pi}_n = (\pi(n,0)\ \ \pi(n,1)\ \ \ldots)$ and $\pi(n,m) = \sum_i c_i \alpha_i^n \beta_i^m$, $n, m > 0$.

Then,

$$\Pi(x,y) = \boldsymbol{\pi}_0 \boldsymbol{y}' + \boldsymbol{\pi}_1 (x^{-1}\boldsymbol{I} - \boldsymbol{R})^{-1} \boldsymbol{y}'$$

where $\boldsymbol{y}' = (1\ y\ y^2\ \ \ldots)$.

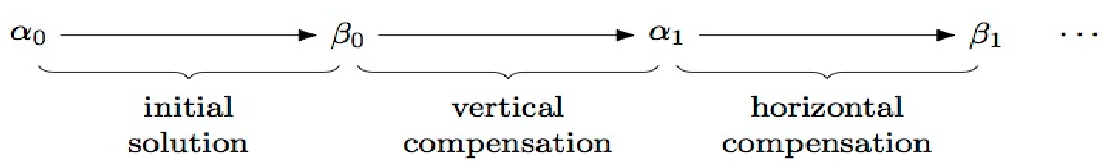
Substituting in the functional equation reveals

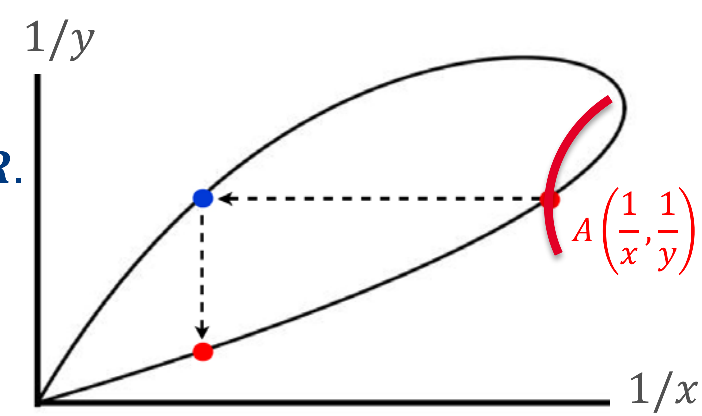
$$K(x,y)\Pi(x,y) = A(x,y)\Pi(x,0) + B(x,y)\Pi(0,y) + C(x,y)\Pi(0,0) \ \ \Rightarrow$$

$$\boldsymbol{\pi}_1 (x^{-1}\boldsymbol{I} - \boldsymbol{R})^{-1}[K(x,y)\boldsymbol{y} + A(x,y)\boldsymbol{e}] \\ = -\boldsymbol{\pi}_0\big[\big(K(x,y) + B(x,y)\big)\boldsymbol{y}' + \big(A(x,y) + C(x,y)\big)\boldsymbol{e}'\big]$$

So $x^{-1} = \alpha$ is an eigenvalue of matrix $\boldsymbol{R}$.

The terms $y^{-1} = \beta$ *are associated with the eigenvalues of* $\boldsymbol{R}$.





Matrix geometric approach

Theorem 1

The terms $\{\alpha_i\}$ constitute the different eigenvalues of the matrix $\boldsymbol{R}$. For eigenvalue α_i the corresponding eigenvector of the matrix $\boldsymbol{R}$ is $\boldsymbol{h_i}$ with $h_{i,m} = c_i(\beta_{i-1}^m + f_i\beta_i^m)$.

Theorem 2

Spectral decomposition

$$\boldsymbol{R} = \boldsymbol{H}^{-1}\boldsymbol{D}\boldsymbol{H}$$

Truncated spectral decomposition

$$\boldsymbol{R_N} = \boldsymbol{H_N^{-1}}\boldsymbol{D_N}\boldsymbol{H_N}$$

Remark

The latter is equivalent to truncating

$$\pi(n,m) = \sum_{i=0}^{N} c_i\alpha_i^n\beta_i^m\,, n, m > 0$$

Main results

Theorem 3

We obtain the eigenvalues of the rate matrix for the original model.

$$K(x,y)\Pi(x,y) = A(x,y)\Pi(x,0) + B(x,y)\Pi(0,y) + C(x,y)\Pi(0,0) + D(x,y)\Pi(p+(1-p)x,y)$$

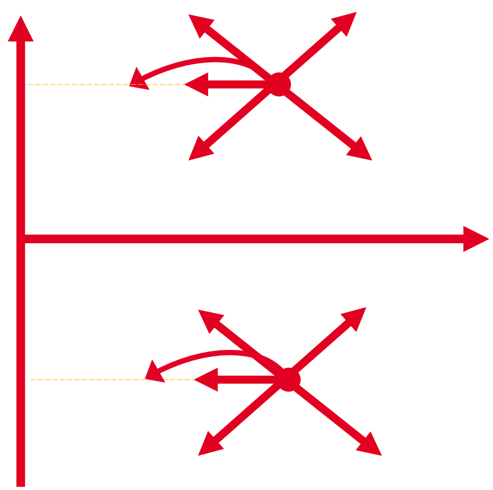
By using a similar argument as previously we obtain

$$\boldsymbol{\alpha_0} \longrightarrow \boldsymbol{p+(1-p)\alpha_0} \longrightarrow \boldsymbol{p^2+(1-p^2)\alpha_0} \longrightarrow$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\boldsymbol{\alpha_1} \longrightarrow \quad \vdots \quad \longrightarrow \quad \vdots$$

$$\downarrow$$



Conclusions

- Calculation of eigenvalues and eigenvectors of rate matrix for NNRW
- Efficient numerical calculation of rate matrix using spectral truncation
- Our results show promise for "non-structured" $\boldsymbol{R}$ of random walks in the quadrant

Extensions

- Probabilistic interpretation of the product-form terms
- Use the results for approximation, i.e. approximate the invariant measure by a series (finite or infinite) of product forms.

  [6] Y. Chen, R.J. Boucherie, and J. Goseling, (2016). Invariant measures and error bounds for random walks in the quarter-plane based on sums of geometric terms, arXiv:1502.07218.