On the Exact Complexity of Hamiltonian Cycle and q-Colouring in Disk Graphs

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Overview

1. Exact complexity and the square root phenomenon
2. Algorithm for Hamiltonian Cycle
3. Lower bound for Hamiltonian Cycle
4. Interesting developments, conclusion
Exact complexity and the square root phenomenon
There is no $2^{o(n)}$ algorithm that solves 3-SAT on $n$ variables.

ETH is great for conditional lower bounds.

**Exact complexity is** $2^{\Theta(f(n))}$ **if:**
- algorithm runs in $2^{O(f(n))}$
- $2^{o(f(n))}$ would contradict ETH
Unit Disk Graph (UDG):
- vertex set: unit disks in the Euclidean plane
- edges: between intersecting pairs of unit disks

Given by its representation (coordinates of disk centers).
The “square root” phenomenon

Credit to Cai, Juedes, Marx, Sidiropoulos and many others.

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<th>Best algorithm (ETH)</th>
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- Two cells are *neighbors* if there is an edge between them.

How many times should a Hamiltonian Cycle alternate between two neighboring cells?

**Answer:** (by Ito & Kadoshita, 2010)

At most constant times.

With a treewidth based Ham. Cycle algorithm, we get $2^{O(\sqrt{n})}$ time.
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![Grid diagram](image)
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Lower bound for Hamiltonian Cycle
Starting out

Φ is a 3-SAT formula on \( n \) variables and \( m \) clauses. We modify a classical directed HC reduction (by Kleinberg and Tardos):

\[ \Phi := (x_1 \lor \bar{x}_2 \lor x_3) \land (x_2 \lor \bar{x}_3 \lor \bar{x}_4). \]
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- Get from \( v_{start} \) to \( v_{end} \) by going through each variable row in some direction (left to right = TRUE)
- Walk around each clause loop on the path of one of its true literals.
... and fix a drawing of the undirected version

By changing each vertex to an in- mid- and out-vertex, we get $G$.

$G$ has an *undirected* Hamiltonian Cycle iff $\Phi$ is satisfiable.
We cannot just change edges to chains of unit disks.
Edges to threads and snakes

Note

We cannot just change edges to chains of unit disks.

We use *snakes* instead, inspired by Itai et al. (1982).
We had trouble with two crossing snakes.

**Thread:** chain of unit disks
Crossings

We had trouble with two crossing snakes.

**Thread:** chain of unit disks

We can cross a snake and a thread.

Luckily that’s all we need.
Threads

If $\deg(v) = 2$:
Edges incident to $v$ are present in all Hamiltonian Cycles.

Such edges can be replaced by threads.
After some adjustments around vertices:

Fig: A variable’s row crosses a clause loop.
Final construction

**Theorem**
There is a HC in the construction iff $G$ has a HC iff $\Phi$ is satisfiable.

**Theorem**
The construction fits in a box of size $O(n + m) \times O(n + m)$, and uses $O((n + m)^2)$ unit disks.

Wlog. $m = \Theta(n)$, so a $2^{o(\sqrt{|V|})}$ algorithm would violate ETH.
In the meantime...
Hamiltonian Cycle in grids is NP-complete, and (Itai et al. (1982)) implies a $2^\Omega(\sqrt[3]{n})$ lower bound.

**Theorem**

*The best algorithm for Hamiltonian Cycle in grid graphs under ETH has running time $2^{\Theta(\sqrt{n})}$.*

The gadgetry requires more finesse. (Bipartiteness!)
Higher dimensions, other problems, conclusion

Ongoing work includes:

- In $d$ dimensions the running time should be $2^{\Theta(n^{1-1/d})}$ for unit balls.
- Coloring: for $\ell = n^\alpha$ colors, $2^{\tilde{\Theta}(\sqrt{n\ell})}$ in 2 dimensions and $2^{\tilde{\Theta}(n^{1-1/d}\ell^{1/d})}$ generally by Biró et al. (2017)
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Open:

- Algorithms in disk and ball graphs of arbitrary radii, e.g. for Hamiltonian Cycle? (Independent set and Coloring behave well)
- Can we get by without a representation?
ETH says $2^{\Theta(\sqrt{n})!}$

Thank you! Questions?