Chapter 5

Rotations and projections

In this chapter we discuss

- Rotations
- Parallel and perspective projections such as used in representing 3d images.

Using coordinates and matrices, parallel projections and rotations can be described explicitly in such a way that computers can handle them. This is at the basis of computer generated pictures.

5.1 Rotations

5.1.1 Combining rotations around coordinate axes in 3d

A rotation in 3-space involves an axis around which to rotate, and an angle of rotation. But what happens if we combine two rotations? We will discuss combining two rotations around coordinate axes. The general situation is beyond the scope of these notes.

Recall that the rotation around the \( z \)-axis over \( \alpha \) radians (in the positive direction) is written in matrix language as follows. The vector \((x_1, x_2, x_3)\) is multiplied by the rotation matrix:

\[
\begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 \\
\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}.
\]

To rotate the resulting vector, \((x_1 \cos \alpha - x_2 \sin \alpha, x_1 \sin \alpha + x_2 \cos \alpha, x_3)\), around the \( y \)-axis over \( \beta \) radians, multiply the column form of the result with the appropriate rotation matrix:

\[
\begin{pmatrix}
\cos \beta & 0 & -\sin \beta \\
0 & 1 & 0 \\
\sin \beta & 0 & \cos \beta
\end{pmatrix}
\begin{pmatrix}
x_1 \cos \alpha - x_2 \sin \alpha \\
x_1 \sin \alpha + x_2 \cos \alpha \\
x_3
\end{pmatrix}.
\]
If you work this out, you end up with a considerable expression. But thanks to our matrix notation, we can streamline the result in the following way.

\[
\begin{pmatrix}
\cos \alpha \cos \beta & -\sin \alpha \cos \beta & -\sin \beta \\
\sin \alpha & \cos \alpha & 0 \\
\cos \alpha \sin \beta & -\sin \alpha \sin \beta & \cos \beta
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}.
\]

The matrix on the left is called the *product matrix* of the matrices

\[
\begin{pmatrix}
\cos \beta & 0 & -\sin \beta \\
0 & 1 & 0 \\
\sin \beta & 0 & \cos \beta
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 \\
\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

### 5.1.2 Intermezzo: The product of two matrices

It is about time we say a bit more on matrix multiplication. To begin with, two matrices $A$ and $B$ can be multiplied in this order precisely if the rows of $A$ have the same length as the columns of $B$. If this is the case, then the elements of the product $AB$ are obtained by taking appropriate dot products of rows of $A$ and columns of $B$. This way of multiplying two matrices may seem weird at first sight, but it turns out to be very useful.

Here are the details. If $A$ is an $m$ by $n$ matrix (rows of length $n$) and $B$ is an $n$ by $p$ matrix (columns of length $n$), then $AB$ will be an $m$ by $p$ matrix. In row $i$ and column $j$ (spot $i,j$ for short) of the product $AB$ (where $1 \leq i \leq m$ and $1 \leq j \leq p$), we find the dot product of the $i$-th row of $A$ and the $j$-th column of $B$. To digest this, we take a look at a few examples.

- Take

\[
A = \begin{pmatrix} 1 & 8 \\ 2 & -3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 7 & -2 \\ 3 & -1 & 5 \end{pmatrix}.
\]

So $A$ is a 2 by 2 matrix and $B$ is a 2 by 3 matrix. The two rows of $A$ contain two elements each, just like the three columns of $B$. So the matrices can be multiplied. The product matrix $AB$ will be a 2 by 3 matrix. To compute the entry in position 2, 3 we need the dot product of the 2nd row of $A$ and the 3rd column of $B$ (these are printed in bold face). So this entry is $\begin{pmatrix} 2 & -3 \end{pmatrix} \cdot \begin{pmatrix} -2, 5 \end{pmatrix} = -19$. Likewise, the element in position 1, 1 is $(1, 8) \cdot (2, 3) = 26$, etc. The total result is

\[
AB = \begin{pmatrix} 1 & 8 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 2 & 7 & -2 \\ 3 & -1 & 5 \end{pmatrix} = \begin{pmatrix} 26 & -1 & 38 \\ -5 & 17 & -19 \end{pmatrix}.
\]

- The product of $B$ and $A$ does not exist, since the rows of $B$ have length 3, whereas the columns of $A$ have length 2.
• Let us look at the product of the two rotation matrices
\[
\begin{pmatrix}
\cos \beta & 0 & -\sin \beta \\
0 & 1 & 0 \\
\sin \beta & 0 & \cos \beta
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 \\
\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]
Both matrices are 3 by 3 matrices, so the product of the two does exist. If we multiply the matrices in the given order, then the element in position 1,1 of the resulting product matrix is
\[(\cos \beta, 0, -\sin \beta) \bullet (\cos \alpha, \sin \alpha, 0) = \cos \beta \cos \alpha.
\]
In this way the whole matrix product can be computed. We get
\[
\begin{pmatrix}
\cos \alpha \cos \beta & -\sin \alpha \cos \beta & -\sin \beta \\
\sin \alpha & \cos \alpha & 0 \\
\cos \alpha \sin \beta & -\sin \alpha \sin \beta & \cos \beta
\end{pmatrix}.
\]
If we change the order of the two matrices, then the matrix product still exists, but the resulting product is in general a different matrix. The product of
\[
\begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 \\
\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
and \(\begin{pmatrix}
\cos \beta & 0 & -\sin \beta \\
0 & 1 & 0 \\
\sin \beta & 0 & \cos \beta
\end{pmatrix}\) is
\[
\begin{pmatrix}
\cos \alpha \cos \beta & -\sin \alpha & -\cos \alpha \sin \beta \\
\sin \alpha \cos \beta & \cos \alpha & -\sin \alpha \sin \beta \\
\sin \beta & 0 & \cos \beta
\end{pmatrix}.
\]
Take \(\alpha = \beta = \pi/4\) and look at the spots 2,3 and 3,2 to see that this matrix differs from the previous product.

• An important situation for us occurs when \(A\) is a 3 by 3 matrix and \(B\) is a 3 by 1 matrix. The product \(AB\) is again a 3 by 1 matrix. For instance,
\[
\begin{pmatrix}
2 & 3 & -1 \\
6 & -2 & 3 \\
-5 & 2 & 8
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} = \begin{pmatrix}
2x_1 + 3x_2 - x_3 \\
6x_1 - 2x_2 + 3x_3 \\
-5x_1 + 2x_2 + 8x_3
\end{pmatrix}.
\]
Geometric transformations like rotations around an axis through the origin and reflections in a plane through the origin can be described by 3 by 3 matrices (for rotations around a coordinate axis we have already seen this). The effect of the transformation on a vector is then easily computed through a matrix multiplication.

5.1.3 Intermezzo: Scalar multiplication of matrices and a linearity property

Multiplying a vector by a scalar means that all coordinates of the vector are multiplied by that scalar. Similarly, multiplying a matrix by a scalar means that all elements of that matrix are multiplied by that scalar; this is called the scalar product. We denote this as
follows: if $A$ is a matrix, then the scalar product of $\lambda$ and $A$ is written as $\lambda A$ or $\lambda \cdot A$. Here is an example:

\[ 3 \cdot \begin{pmatrix} 2 & 5 & 11 \\ 7 & -4 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 15 & 33 \\ 21 & -12 & 0 \end{pmatrix}. \]

This scalar product is also used to ‘extract’ a common factor in order to get a simpler looking expression. For instance,

\[ \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} = \frac{\sqrt{2}}{2} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \]

If $A$ is a 3 by 3 matrix and $p + \lambda v$ is a line, then, using the appropriate column form of the vectors, it is easily verified (but a somewhat boring computation, so we skip it) that

\[ A(p + \lambda v) = Ap + \lambda Av. \]

This equality has an interesting interpretation. If $A$ represents a geometric transformation like a rotation, then this equality says that if you transform a line, the end result is again a line, now with support vector $Ap$ and direction vector $Av$. There is one exception: if $Av = 0$, then the end result consists of just one vector. This can happen in the case of projections.

### 5.1.4 General rotations around an axis through the origin

Using suitable matrix products, a matrix can be computed for any rotation around an axis through the origin. Since this will involve more geometric considerations, we refrain from going into details here.

### 5.2 Projections

#### 5.2.1 The orthogonal projection on a line through the origin

You are probably familiar with decomposing a force in mutually perpendicular components. Here, we describe the closely related subject of projecting a vector on a line through the origin. Suppose the line $\ell$ is spanned by the vector $a = (1, 2, 2)$ and suppose we wish to decompose the vector $(2, 5, 3)$ in a component (vector) along $\ell$ and a component perpendicular to $\ell$, i.e., we wish to find two vectors $u$ and $v$ such that

- $u$ is on $\ell$, so $u = \lambda(1, 2, 2)$ for some $\lambda$. So the real problem is to find $\lambda$. There are two more conditions that can help us in determining $\lambda$.
- $u$ and $v$ are perpendicular, so $u \cdot v = 0$.
- The sum of $u$ and $v$ equals $(2, 5, 3)$, i.e., $u + v = (2, 5, 3)$.

From the first and the third condition, we conclude that

\[ v = (2, 5, 3) - u = (2, 5, 3) - \lambda(1, 2, 2) = (2 - \lambda, 5 - 2\lambda, 3 - 2\lambda). \]
5.2 Projections

Figure 5.1: The orthogonal projection of $\mathbf{x}$ on the line $\mu \mathbf{a}$. We look for a vector $\lambda \mathbf{a}$ such that $\mathbf{x} - \lambda \mathbf{a}$ is perpendicular to the line, i.e., perpendicular to $\mathbf{a}$.

Using the second condition we can determine $\lambda$ and hence $\mathbf{u}$ and $\mathbf{v}$:

$$0 = \mathbf{a} \cdot \mathbf{v} = (1, 2, 2) \cdot (2 - \lambda, 5 - 2\lambda, 3 - 2\lambda) = (2 - \lambda) + 2(5 - 2\lambda) + 2(3 - 2\lambda) = 18 - 9\lambda.$$

So $\lambda = 2$. With this value of $\lambda$, it follows that $\mathbf{u} = (2, 4, 4)$ and $\mathbf{v} = (0, 1, -1)$.

In the general case, the computation is as follows. Suppose we want to have the orthogonal projection of $\mathbf{x}$ on the line $\mu \mathbf{a}$. This projection is of the form $\mathbf{u} = \lambda \mathbf{a}$ for some $\lambda$. So we need to find $\lambda$. Since $\mathbf{x} - \mathbf{u}$ is perpendicular to $\mathbf{a}$, we get the following equation for $\lambda$:

$$0 = \mathbf{a} \cdot (\mathbf{x} - \lambda \mathbf{a}) = \mathbf{a} \cdot \mathbf{x} - \lambda \mathbf{a} \cdot \mathbf{a}.$$

Solving for $\lambda$ gives us $\lambda = \frac{\mathbf{a} \cdot \mathbf{x}}{\mathbf{a} \cdot \mathbf{a}}$. The projection $\mathbf{u}$ is therefore

$$\frac{\mathbf{a} \cdot \mathbf{x}}{\mathbf{a} \cdot \mathbf{a}} \mathbf{a}.$$

The projection is simply $(\mathbf{a} \cdot \mathbf{x}) \mathbf{a}$ if $\mathbf{a}$ has length 1, i.e., $|\mathbf{a}| = 1$ and so $\mathbf{a} \cdot \mathbf{a} = 1$.

For example, to project $(2, 3, 0)$ on the line $\lambda(1, 0, 1)$, we compute $|(1, 0, 1)| = \sqrt{2}$ and the dot product $(2, 3, 0) \cdot (1, 0, 1) = 2$. So the projection is

$$\frac{2}{\sqrt{2}} \cdot (1, 0, 1) = (\sqrt{2}, 0, \sqrt{2}).$$

5.2.2 The orthogonal projection on a plane through the origin

In a similar way the orthogonal projection onto a plane can be computed. If the plane $V$ is $\lambda \mathbf{a} + \mu \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are mutually perpendicular, then the orthogonal projection of the vector $\mathbf{x}$ on $V$ is given by

$$\frac{\mathbf{a} \cdot \mathbf{x}}{\mathbf{a} \cdot \mathbf{a}} \mathbf{a} + \frac{\mathbf{b} \cdot \mathbf{x}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b}.$$

Note the condition on $\mathbf{a}$ and $\mathbf{b}$: they must be perpendicular (if they are not perpendicular, it is a bit harder to describe the projection explicitly). The formula for the projection
simplifies if \( \mathbf{a} \) and \( \mathbf{b} \) have length 1. If \( |\mathbf{a}| = |\mathbf{b}| = 1 \) (and \( \mathbf{a} \cdot \mathbf{b} = 0 \)), then the projection of \( \mathbf{x} \) on the plane \( \lambda \mathbf{a} + \mu \mathbf{b} \) is \( (\mathbf{a} \cdot \mathbf{x}) \mathbf{a} + (\mathbf{b} \cdot \mathbf{x}) \mathbf{b} \).

For example, let us compute the projection of \((2, 5, 6)\) on the plane \( V : \lambda(1, 0, 1) + \mu(2, 1, -2) \). Here are the steps to take.

- First we check that the vectors \( \mathbf{a} = (1, 0, 1) \) and \( \mathbf{b} = (2, 1, -2) \) are perpendicular:
  \[
  (1, 0, 1) \cdot (2, 1, -2) = 1 \cdot 2 + 0 \cdot 1 + 1 \cdot (-2) = 0.
  \]

- Then we compute \( |\mathbf{a}| = \sqrt{2} \) and \( |\mathbf{b}| = 3 \). We also need the inner products \( \mathbf{a} \cdot (2, 5, 6) = 8 \) and \( \mathbf{b} \cdot (2, 5, 6) = -3 \).

- The projection is therefore
  \[
  \frac{8}{\sqrt{2}} \cdot (1, 0, 1) + \frac{-3}{3} \cdot (2, 1, -2) = (4\sqrt{2}, 0, 4\sqrt{2}) - (2, 1, -2) = (4\sqrt{2} - 2, -1, 4\sqrt{2} + 2).
  \]

### 5.2.3 Example: computing the projection without using the formula

The formula for the projection onto a plane will be useful when we discuss parallel projections. Since the formula does not work if the vectors spanning the plane are not perpendicular, we illustrate how to compute the projection of a single vector directly. Suppose

![Diagram](image-url)

**Figure 5.2:** The orthogonal projection of \( \mathbf{x} \) on the plane spanned by \( \mathbf{a} \) and \( \mathbf{b} \). In this case we are looking for a vector \( \mathbf{c} = u\mathbf{a} + v\mathbf{b} \) in the plane such that \( \mathbf{x} - \mathbf{c} \) is perpendicular to the plane.

the plane on which we project is \( V : \lambda(1, 2, 3) + \mu(2, -1, 1) \). We wish to find the projection of \( \mathbf{x} = (4, 2, 3) \) on the plane. Here are the steps to be taken.

- We wish to find a vector in the plane, say \( \mathbf{c} = u(1, 2, 3) + v(2, -1, 1) \) such that \( \mathbf{x} - \mathbf{c} \) is perpendicular to the plane. So \( \mathbf{x} - \mathbf{c} \) should have dot product 0 with both \((1, 2, 3)\) and \((2, -1, 1)\).
The conditions \((\mathbf{x} - \mathbf{c}) \cdot (1, 2, 3) = 0\) and \((\mathbf{x} - \mathbf{c}) \cdot (2, -1, 1) = 0\) give us two equations from which we have to deduce \(u\) and \(v\). Since \(\mathbf{x} - \mathbf{c} = (4 - u - 2v, 2 - 2u + v, 3 - 3u - v)\) we get

\[
\begin{align*}
1 \cdot (4 - u - 2v) + 2 \cdot (2 - 2u + v) + 3 \cdot (3 - 3u - v) &= 0 \\
2 \cdot (4 - u - 2v) - 1 \cdot (2 - 2u + v) + 1 \cdot (3 - 3u - v) &= 0.
\end{align*}
\]

After rearranging we obtain the equations

\[
\begin{align*}
-14u - 3v &= -17 \\
-3u - 6v &= -9,
\end{align*}
\]

with solution \(u = 1\) and \(v = 1\). Consequently, the projection is \((1, 2, 3) + (2, -1, 1) = (3, 1, 4)\).

### 5.2.4 Example: computing the projection using a normal vector

If an equation of the plane is given, then you can also proceed as follows to compute the orthogonal projection of \((4, 2, 3)\). Suppose the plane is \(V : x + y - z = 0\) (this is the same plane as in the previous example). A normal vector is \(\mathbf{n} = (1, 1, -1)\). To compute the orthogonal projection, we start in \((4, 2, 3)\) and move along the line perpendicular to the plane until we reach the point of intersection. So we intersect the line \((4, 2, 3) + \lambda(1, 1, -1)\) with \(V\). To find this intersection, substitute \((4 + \lambda, 2 + \lambda, 3 - \lambda)\) in the equation:

\[(4 + \lambda) + (2 + \lambda) - (3 - \lambda) = 0.
\]

So \(3\lambda + 3 = 0\) and \(\lambda = -1\). Therefore the projection is \((4, 2, 3) - (1, 1, -1) = (3, 1, 4)\) as before.

### 5.2.5 Remark on the orthogonal projection

The orthogonal projection of a vector \(\mathbf{x}\) on a plane (or line) is the vector whose distance to \(\mathbf{x}\) is minimal among all vectors from the plane. This follows from Pythagoras' Theorem.

## 5.3 Parallel projections

### 5.3.1 Various kinds of projections are used in representing 3-dimensional objects in 2d drawings.

Perspective projections probably produce the most natural results, though in technical drawings parallel projections are often used. This section discusses parallel projections and their relation to perspective projections. The focus will be on coordinate methods.

### 5.3.2 Parallel projection: definition

A parallel projection onto a plane \(V\) arises as follows. Fix a direction vector, say \(\mathbf{v}\), in 3-space such that \(\mathbf{v}\) is not parallel to \(V\). For any vector \(\mathbf{p}\) in 3-space, the line \(\mathbf{p} + \lambda \mathbf{v}\) meets \(V\) in a single vector, say \(\mathbf{p}'\). This vector \(\mathbf{p}'\) is the parallel projection of \(\mathbf{p}\) on \(V\). In particular, vectors in \(V\) are projected onto themselves. If the direction vector happens to be perpendicular to the plane \(V\), we say the projection is an orthogonal projection.
5.3.3 Catching parallel projections in coordinates

Let us project onto the $x_2, x_3$-plane, i.e., the plane spanned by the standard basis vectors $e_2$ and $e_3$. Regardless of the direction of projection, the vector $e_2$ projects onto itself as does $e_3$. Now suppose $e_1$ projects to $ae_2 + be_3$, i.e., $(0, a, b)$, for some $a$ and $b$. Geometrically, it is clear that

- $3e_1$ projects to $(0, 3a, 3b)$. More generally, $\lambda e_1$ projects to $(0, \lambda a, \lambda b)$.

- $e_3 + e_1$ projects to $(0, 0, 1) + (0, a, b) = (0, a, b + 1)$. More generally, $e_1 + \lambda e_2 + \mu e_3$ maps to $(0, a, b) + (0, \lambda, 0) + (0, 0, \mu) = (0, a + \lambda, b + \mu)$.

- The vector $(x_1, x_2, x_3)$ projects to $(0, x_1a + x_2, x_1b + x_3)$. To see this, decompose the vector as $(x_1, 0, 0) + (0, x_2, x_3)$ and note that nothing happens with $(0, x_2, x_3)$ and that $(x_1, 0, 0)$ projects to $(0, x_1a, x_1b)$.

We can rewrite this projection in terms of matrices, since

$$
\begin{pmatrix}
0 & 0 & 0 \\
a & 1 & 0 \\
b & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
ax_1 + x_2 \\
bx_1 + x_3
\end{pmatrix}
$$

If we choose $a = b = -0.5$, the matrix looks like

$$
\begin{pmatrix}
0 & 0 & 0 \\
-0.5 & 1 & 0 \\
-0.5 & 0 & 1
\end{pmatrix}
$$

Using this matrix, the projection of any point can be easily computed. For instance, the projection of $(1, 1, 0)$ is $(0, 0.5, -0.5)$. Similarly, the projections of all the vertices of the
5.3 Parallel projections

Figure 5.4: Parallel projection on the $x_2,x_3$–plane is determined by what happens with $e_1$. If $e_1 = (1, 0, 0)$ is projected on $(0, a, b)$ in the $x_2,x_3$–plane, then, for instance, $e_1 + e_3 = (1, 0, 1)$ is mapped to $(0, a, b + 1)$.

Our matrix description is useful if we wish to let a computer generate projections of buildings given the coordinates of those points that determine the shape of the building.

5.3.4 Orthogonal projection

A more natural parallel projection occurs if the direction of projection is perpendicular to the plane onto which we project. For our matrix this means that $a = b = 0$. The resulting matrix is

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

In this case, the matrix is hardly needed anymore for computations, since $(x_1, x_2, x_3)$ is simply projected onto $(0, x_2, x_3)$ (the first coordinate simply ‘vanishes’).

For this orthogonal projection, the standard unit cube is projected onto a square, which is not what we want. In order to get a reasonable picture of a cube, we first rotate it.
Suppose we first rotate over \( \alpha \) (radians or degrees) around the \( x_3 \)-axis, and then over an angle \( \beta \) around the \( x_2 \)-axis. This corresponds to the product of two rotation matrices:

\[
\begin{pmatrix}
\cos \beta & 0 & -\sin \beta \\
0 & 1 & 0 \\
\sin \beta & 0 & \cos \beta
\end{pmatrix}
\begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 \\
\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
\cos \alpha \cos \beta & -\sin \alpha \cos \beta & -\sin \beta \\
\sin \alpha & \cos \alpha & 0 \\
\cos \alpha \sin \beta & -\sin \alpha \sin \beta & \cos \beta
\end{pmatrix}.
\]

In the right-hand matrix, the first column is the result of applying the two rotations to \( e_1 = (1, 0, 0) \), the second column contains the image of \( e_2 \) and the third column contains the image of \( e_3 \). If we apply our orthogonal projection, the first row has to be replaced by zeros, i.e., the vector \( x \) is finally projected to

\[
\begin{pmatrix}
0 & 0 & 0 \\
\sin \alpha & \cos \alpha & 0 \\
\cos \alpha \sin \beta & -\sin \alpha \sin \beta & \cos \beta
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_2 \\ x_3
\end{pmatrix}.
\]

Varying the angles \( \alpha \) and \( \beta \), projections adapted to one’s needs can be produced.

We note one important property of our projection (matrix). Let us call the columns of the matrix \( \mathbf{a}_1, \mathbf{a}_2 \) and \( \mathbf{a}_3 \). Then

\[
|\mathbf{a}_1|^2 + |\mathbf{a}_2|^2 + |\mathbf{a}_3|^2 = 2,
\]

since

\[
\sin^2 \alpha + (\cos \alpha \sin \beta)^2 + \cos^2 \alpha + (\sin \alpha \sin \beta)^2 + \cos^2 \beta = \\
\sin^2 \alpha + \cos^2 \alpha + \sin^2 \beta(\cos^2 \alpha + \sin^2 \alpha) + \cos^2 \beta = \\
1 + \sin^2 \beta + \cos^2 \beta = 2.
\]

We distinguish two special cases, the \textit{isometric projection} and the \textit{dimetric projection}. 

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Figure 5.5: \textit{Parallel projection of a cube.} The standard unit cube on the left-hand side is projected on the \( x_2, x_3 \)-plane. The direction is determined by what happens to \( e_1 \) (indicated by the dashed line). In the middle the parallel projections of the front and the back of the cube have been drawn. The picture on the right-hand side has been computed from the coordinates of the standard unit cube and the projection matrix with \( a = b = -1/2 \).
• If \(|a_1| = |a_2| = |a_3|\), the projection is called isometric. In this case, \(|a_1| = |a_2| = |a_3| = \sqrt{2}/3\). From the third column we deduce \(\cos \beta = \pm \sqrt{2}/3\) (and so \(\sin^2 \beta = 1 - 2/3 = 1/3\)). The first two columns lead to the equations

\[
\begin{align*}
\sin^2 \alpha + \cos^2 \alpha/3 &= 2/3 \\
\cos^2 \alpha + \sin^2 \alpha/3 &= 2/3
\end{align*}
\]

This means that \(\cos \alpha = \pm \sqrt{2}/2\) and \(\sin \alpha = \pm \sqrt{2}/2\). From all the possible solutions, let us take the one with \(\sin \alpha = -\sqrt{2}/2\), \(\cos \alpha = \sqrt{2}/2\) \((\alpha = -45^\circ)\), \(\cos \beta = \sqrt{2}/3\) and \(\sin \beta = -\sqrt{1/3}\). This gives the following vectors in the projection plane

<table>
<thead>
<tr>
<th>point of cube</th>
<th>coordinates of projection</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 0, 0)</td>
<td>((-\sqrt{2}/2, -1/\sqrt{6}) \approx (-0.71, -0.41))</td>
</tr>
<tr>
<td>(0, 1, 0)</td>
<td>((\sqrt{2}/2, -1/\sqrt{6}) \approx (0.71, -0.41))</td>
</tr>
<tr>
<td>(0, 0, 1)</td>
<td>((0, \sqrt{2}/3) \approx (0, 0.82))</td>
</tr>
</tbody>
</table>

Figure 5.6: *In the isometric projection, the three edges of the cube connecting (0, 0, 0) to (1, 0, 0), (0, 1, 0) and (0, 0, 1), respectively, are projected to three symmetrically distributed axes in the projection plane (left-hand picture). The projection of the standard unit cube is shown on the right. In the case of the isometric projection, the three major axes of the cube are foreshortened equally.*

• If the ratio of the three lengths is 1 : 2 : 2 the projection is called a dimetric projection. In this case the foreshortening of the three main axes of the cube is not equal. An example of a dimetric projection matrix is

\[
\begin{pmatrix}
0 & 0 & 0 \\
-\sqrt{1/8} & \sqrt{7/8} & 0 \\
-\sqrt{7/72} & -\sqrt{1/72} & \sqrt{8/9}
\end{pmatrix}
\]

The entries have been computed by solving a similar pair of equations as in the isometric case.
Figure 5.7: In the dimetric projection in the figure, the rotating and projecting the three edges of the cube connecting $(0, 0, 0)$ to $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, respectively, leads to a foreshortening in the ratio $1:2:2$. The resulting projection of the standard unit cube is shown on the right.

5.3.5 Orthogonal projection: a different perspective

Instead of rotating the object to get a projection in which the object looks better, we can also rotate the plane on which we project. This has the same end result, and may be more practical (buildings cannot be rotated so easily!). But the mathematics is slightly different.

So suppose we rotate the $x_2, x_3$-plane around the $x_3$-axis over the angle $\alpha$ and around the $x_2$-axis over the angle $\beta$, then the rotated plane $V$ is spanned by the images of $e_2$ and $e_3$. We call them $b_2$ and $b_3$, respectively. So $V : \lambda b_2 + \mu b_3$. Note that the vectors $b_2$ and $b_3$ have length 1 and are perpendicular. Therefore we can apply the projection formula of the previous section. The orthogonal projection on the plane $V$ of a vector $x$ is computed as follows:

$$(x \cdot b_2) b_2 + (x \cdot b_3) b_3.$$

So $x \cdot b_2$ units along the $b_2$-axis and $x \cdot b_3$ units along the $b_3$-axis. In particular, the orthogonal projection of $e_1$ can be computed with this formula: $(e_1 \cdot b_2) b_2 + (e_1 \cdot b_3) b_3$. We call this third vector $b_1$.

If this plane $V$ represents the paper we draw our projection on, then this is what happens.

- The coordinate system is the one with axes along $b_2$ and $b_3$, respectively. In this coordinate system, the projection of $x$ is drawn on the spot with coordinates

$$(x \cdot b_2, x \cdot b_3).$$

If you decide to draw the projection of the $x_1$-axis in 3-space, then draw the line through $(0, 0)$ and $(e_1 \cdot b_2, e_1 \cdot b_3)$.

- In fact, if you want to make a drawing of the projection on any piece of paper, just take a cartesian coordinate system on this paper. To represent the projection of $x$, draw the point

$$(x \cdot b_2, x \cdot b_3)$$
as before.

Here is an example. We rotate the projection plane using \( \alpha = 45^\circ \) and \( \beta = 45^\circ \). The corresponding rotation matrix is

\[
\begin{pmatrix}
    \cos 45^\circ \cos 45^\circ & -\sin 45^\circ \cos 45^\circ & -\sin 45^\circ \\
    \sin 45^\circ & \cos 45^\circ & 0 \\
    \cos 45^\circ \sin 45^\circ & -\sin 45^\circ \sin 45^\circ & \cos 45^\circ
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
    1 & -1 & -\sqrt{2} \\
    \sqrt{2} & \sqrt{2} & 0 \\
    1 & -1 & \sqrt{2}
\end{pmatrix}.
\]

The projection plane is spanned by the 2nd and 3rd column (note the factor 1/2): \( b_2 = (-1/2, \sqrt{2}/2, -1/2) \) and \( b_3 = (-\sqrt{2}/2, 0, \sqrt{2}/2) \). To find the coordinates of the projection of a vector, say the vertex \((1, 0, 0)\) of the standard unit cube, we compute the dot products of \((1, 0, 0)\) with \( b_2 \) and \( b_3 \):

\[
(1, 0, 0) \cdot (-1/2, \sqrt{2}/2, -1/2) = -1/2, \quad (1, 0, 0) \cdot (-\sqrt{2}/2, 0, \sqrt{2}/2) = -\sqrt{2}/2.
\]

So \((1, 0, 0)\) projects to \((-1/2, -\sqrt{2}/2)\) (we have omitted the first coordinate). Here is what happens to the vertices of the cube.

<table>
<thead>
<tr>
<th>vertex of cube</th>
<th>projection</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0, 0))</td>
<td>((0, 0))</td>
</tr>
<tr>
<td>((1, 0, 0))</td>
<td>((-1/2, -\sqrt{2}/2) \approx (-0.5, -0.71))</td>
</tr>
<tr>
<td>((1, 1, 0))</td>
<td>((5\sqrt{2}/2, 0) \approx (0.71, 0))</td>
</tr>
<tr>
<td>((0, 0, 1))</td>
<td>((-1/2, \sqrt{2}/2) \approx (-0.5, 0.71))</td>
</tr>
<tr>
<td>((1, 0, 1))</td>
<td>((-1, 0))</td>
</tr>
<tr>
<td>((1, 1, 1))</td>
<td>((-1 + \sqrt{2}/2, 0) \approx (-0.29, 0))</td>
</tr>
<tr>
<td>((0, 1, 1))</td>
<td>((\sqrt{2}/2, 1/2) \approx (0.21, 0.71))</td>
</tr>
</tbody>
</table>

### 5.3.6 Perspective projections and its relation to parallel projections

The most natural projection is probably the perspective projection. As we will show the orthogonal parallel projection is a kind of limit situation of the perspective projection.

In perspective projection, we are dealing with a situation like the following. We take a point on the \(x_1\)-axis, say \( P = (f, 0, 0) \), called the center of projection. Now we project points on the \(x_2, x_3\)-plane (the view plane): for a point \( Q \) in 3-space, intersect the line \( PQ \) with the \(x_2, x_3\)-plane. The point of intersection is the projection of \( Q \). Using similar triangles, we can express this projection in terms of coordinates:

\[
(x, y, z) \mapsto \left(0, \frac{fy}{f - x}, \frac{fz}{f - x}\right).
\]

If we take the limit as \( f \to \infty \), the expression on the right-hand side simplifies to the expression for the orthogonal projection: \((0, y, z)\). Note that the expression for the projection is not linear since the variables occur in both numerator and denominator. (There is still a way to describe this projection with matrices, but we will not discuss that here.)

We show two approaches to find the explicit expressions for the coordinates.
Figure 5.8: The standard unit cube projected to the plane spanned by $\mathbf{b}_2 = (-1/2, \sqrt{2}/2, -1/2)$ and $\mathbf{b}_3 = (-\sqrt{2}/2, 0, \sqrt{2}/2)$. The $x$–axis is the axis along $\mathbf{b}_2$ and the $y$–axis is the axis along $\mathbf{b}_3$.

- The line $\ell$ connecting $P = (f, 0, 0)$ and $Q = (x, y, z)$ has the following parametric description:

$$\mathbf{r}(\lambda) = (f, 0, 0) + \lambda((x, y, z) - (f, 0, 0)).$$

The first coordinate is therefore $f + \lambda(x - f)$. To find the intersection with the $x_2,x_3$–plane, we need to set the first coordinate equal to 0, i.e., solve $f + \lambda(x - f) = 0$. This leads to $\lambda = \frac{f}{f-x}$. Substituting this value of $\lambda$ in the parametric description yields:

$$(0, \frac{fy}{f-x}, \frac{fz}{f-x}).$$

Figure 5.9: Using similar triangles, the coordinates of the perspective projection $Q'$ of $Q$ on the $x_2,x_3$–plane can be computed. The triangles $\triangle PQA$ and $\triangle PQ'A'$ are similar as well as the triangles $\triangle PBA$ and $\triangle PB'A'$.

- Here is a computation involving similar triangles. Suppose $Q = (x, y, z)$ and suppose the projection has coordinates $Q' = (0, y', z')$. The triangles $\triangle PQA$ and $\triangle PQ'A'$
are similar so
\[
\frac{z}{z'} = \frac{QA}{Q'A'} = \frac{PA}{PA'}.
\]
The triangles \(\triangle PBA\) and \(\triangle PB'A'\) are also similar, hence
\[
\frac{PA}{PA'} = \frac{PB}{PB'} = \frac{f - x}{f}.
\]
So \(\frac{z}{z'} = \frac{f - x}{f}\). Therefore \(z' = \frac{fz}{f - x}\). For the conscientious reader: yes, we have been a bit sloppy with the signs in the ratios.
5.4 Exercises

1 The matrix product
In each of the following cases compute the matrix product $AB$ if it exists. What about $BA$?
   a) $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 4 & -1 & 1 \\ 2 & 1 & 5 & 3 \end{pmatrix}$.
   b) $A = \begin{pmatrix} 2 & 2 \\ 1 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix}$.
   c) $A = \begin{pmatrix} 1 & a & -2 \\ 2 & -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & -2 \end{pmatrix}$.

2 Rotations
   a) Compute the matrix that describes the following combination of rotations: a rotation around the $x_3$-axis in the positive direction over $30^\circ$ followed by a rotation around the $x_2$-axis in the positive direction over $60^\circ$. Compute what happens with $(1, 1, 1)$.
   b) Reverse the order of the two rotations in the previous question. Does anything change?

3 Projection on a line
   a) Determine the (orthogonal) projection of $x = (2, 5, 2)$ on the line $\ell : \mu(1, 1, 1)$ in two ways:
      - by using the appropriate formula;
      - by analysing the geometry of the situation and determining a vector $c$ on $\ell$ such that $x - c$ is perpendicular to $\ell$.
   b) Use a) to determine the distance between $(2, 5, 3)$ and $\ell$.
   c) Which multiple of $x$ projects exactly to $(1, 1, 1)$?

4 Projection on a line Given two non-zero vectors $a$ and $b$ such that $|a| > |b|$. Which one is longer: the orthogonal projection of $a$ on the line $\lambda b$ or the orthogonal projection of $b$ on the line $\mu a$?

5 Projections and shortest distance
Given a plane $V$ and a vector $x$ not in $V$. The orthogonal projection of $x$ on $V$ is denoted by $x'$. Use the triangle in Figure 5.10 to show that the distance between $x$ and $x'$ is less than the distance between $x$ and any other vector $y$ in $V$. What is the relation between $|x - x'|$, $|x - y|$ and $|x' - y|$?

6 Parallel projections of the cube
Refer to (5.3.3) on page 124. For parallel projection on the $x_2, x_3$-plane, the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ can be used to compute the projection of any given vector or point.
Figure 5.10: The distance between \( \mathbf{x} \) and \( \mathbf{x}' \) is less than the distance between \( \mathbf{x} \) and \( \mathbf{y} \).

a) Draw the projection of the standard unit cube (vertices \((0,0,0), (1,0,0), (1,1,0), \) etc.) for \( a = b = -1/2 \).

b) Same question, now for \( a = -2, b = -1 \).

7 On the isometric projection
In Figure 5.6 the (projections of) the three standard basis vectors are drawn: \((-1/\sqrt{2}, -1/\sqrt{6}), (1/\sqrt{2}, -1/\sqrt{6})\) and \((0, \sqrt{2}/3)\). Verify that these vectors make equal angles with each other.

8 On the dimetric projection
Show that to get a dimetric projection matrix as on page 127 you need to solve the system of equations

\[
\begin{align*}
9 \cos^2 \beta &= 8 \\
9 \sin^2 \alpha + \cos^2 \alpha &= 2 \\
9 \cos^2 \alpha + \sin^2 \alpha &= 8.
\end{align*}
\]

9 Turning the plane on which we project
a) Suppose we rotate the \( x_2, x_3 \)-plane over 45° around the \( x_3 \)-axis (in the positive direction). Show that \( \mathbf{e}_2 \) rotates to \( \mathbf{b}_2 = (-\sqrt{2}/2, \sqrt{2}/2, 0) \) and \( \mathbf{e}_3 \) remains unchanged.

b) We use the plane \( V \) spanned by \( \mathbf{b}_2 \) and \( \mathbf{e}_3 \) to project on. What is the projection of \( \mathbf{e}_1 \)? Draw the three axes.

c) Now rotate the \( x_2, x_3 \)-plane over 45° around the \( x_3 \)-axis in the positive direction and then rotate it around the \( x_2 \)-axis over 45° in the positive direction. Compute the 3 by 3 matrix describing this combination of rotations. Show that \( \mathbf{e}_2 \) rotates to \( \mathbf{b}_2 = (-1/2, \sqrt{2}/2, -1/2) \), and \( \mathbf{e}_3 \) rotates to \( \mathbf{b}_3 = (-\sqrt{2}/2, 0, \sqrt{2}/2) \). We use the resulting plane \( W \) spanned by \( \mathbf{b}_2 \) and \( \mathbf{b}_3 \) to project on. What is the projection of \( \mathbf{e}_1 \)? Draw the three axes in \( W \).

10 Perspective projections
Consider the perspective projection from the center \( P = (1, 0, 2) \) on the view plane \( x_1 = 0 \) (the \( x_2, x_3 \)-plane).
a) Sketch the situation. In particular, draw the two lines \( \ell : (0,1,0) + \lambda(1,0,0) \) and \( m : (0,2,0) + \rho(1,0,0) \). Guess from the picture where the projections of these two lines meet. (Note that the lines themselves do not meet since they are parallel.)

b) Now we turn to the computations. To find the projection of \( \ell \), connect each point of \( \ell \) with \( P \), i.e., the plane \( V_\ell \) containing \( P \) and \( \ell \). Find a parametric description of this plane. Now determine the intersection of \( V_\ell \) with the \( x_2, x_3 \)-plane. Similarly, find a parametric description for the plane \( V_m \) containing \( P \) and \( m \). Determine the intersection of \( V_m \) with the \( x_2, x_3 \)-plane.

c) Use b) to determine the point of intersection of the projections of \( \ell \) and \( m \).
Bibliography


