Faster binary-field multiplication and faster binary-field MACs

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Joint work with Daniel J. Bernstein

The scope of this talk

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The Wegman–Carter construction (1981)

- "Universal" hash function + one-time pad: $h_r(m) \oplus s_n$
- Offers information-theoretic security
- What makes the difference is the hash part

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Binary field MACs:

- Less attractive for hardware designers

GMAC

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- Polynomial evaluation MAC: $m_1 r + m_2 r^2 + \cdots$
- Based on arithmetic in

$$\mathbb{F}_{2^{128}} = \mathbb{F}_2[x]/(x^{128} + x^7 + x^2 + x + 1)$$

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Constant-time software speeds:

reference	platform	`PCLMUQDQ`	cycles per byte
Käsper–Schwabe 2009	Core 2	no	14.40
	Sandy Bridge	no	13.10
Krovetz–Rogaway 2011	Westmere	yes	2.00
Gueron 2013	Sandy Bridge	yes	1.79
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The point of Auth256 is to have **low bit operation count**.

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- Base field $\mathbb{F}_{2^{256}} = \mathbb{F}_2[x]/(f')$

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- A *pseudo-dot-product* MAC:

$$(m_1 + r_1)(m_2 + r_2) + (m_3 + r_3)(m_4 + r_4) + \cdots$$

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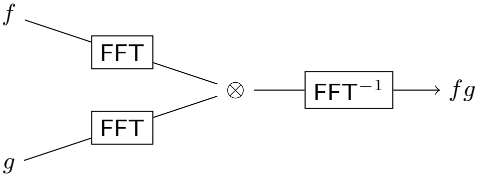
We use **FFT-based** polynomial multiplication.

FFT for polynomial multiplication

- Given polynomials $f, g \in R[x]$, fg is computed by



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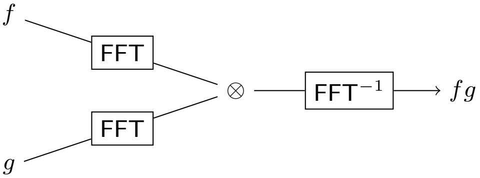

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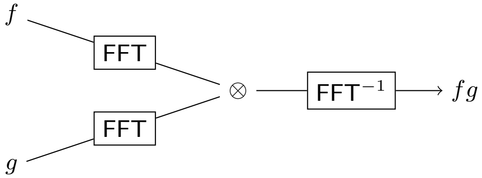

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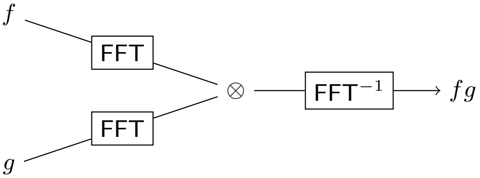

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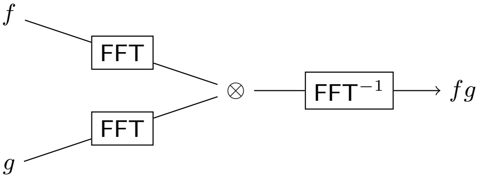

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The design of Auth256 is tailored for **the Gao–Mateer additive FFT** (2010).

Multiplicative FFT

- The key equation (free):

$$f(x) = f^{(0)}(x^2) + x f^{(1)}(x^2)$$

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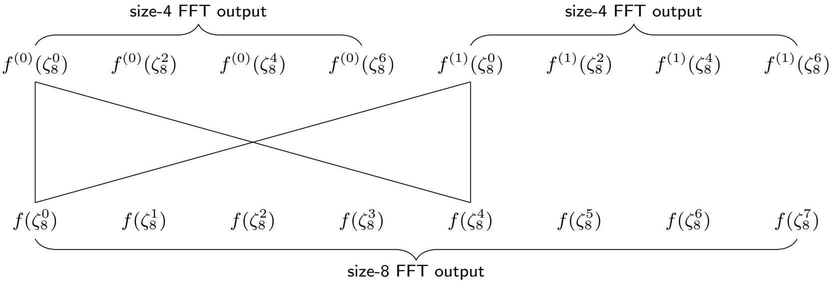
- Size-2^k FFT evaluates f at all 2^k-th roots of unity



The Gao–Mateer additive FFT

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- Size-2^k FFT evaluates f at a k-dimensional $\mathbb{F}_2$-linear subspace in the field

The Gao–Mateer additive FFT (Cont.)

We use a "Cantor basis" for FFTs:

$$\beta_5 \xrightarrow{x^2+x} \beta_4 \xrightarrow{x^2+x} \underbrace{\beta_3 \xrightarrow{x^2+x} \beta_2 \xrightarrow{x^2+x} \underbrace{\beta_1 \xrightarrow{x^2+x} \beta_0 = 1}_{\mathbb{F}_{2^2}}}_{\mathbb{F}_{2^4}}$$

A size-2^k FFT evaluates over the span of $\beta_{k-1}, \ldots, \beta_0$, which means small-size FFTs involve only **multiplications by constants in subfields**.

$\mathbb{F}_{2^8}$ arithmetic: representation

Tower field construction of $\mathbb{F}_{2^8}$:

$$\begin{array}{l}
\mathbb{F}_{2^8} \\
\;\big|\; x_8^2 + x_8 + \alpha_2\alpha_4 \\
\mathbb{F}_{2^4} \\
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$$

Representation of field elements:

- $b_0 + b_1\alpha_2 \in \mathbb{F}_{2^2}$
- $(b_0 + b_1\alpha_2) + (b_2 + b_3\alpha_2)\alpha_4 \in \mathbb{F}_{2^4}$
- $(b_4 + b_5\alpha_2) + (b_6 + b_7\alpha_2)\alpha_4 + ((b_4 + b_5\alpha_2) + (b_6 + b_7\alpha_2)\alpha_4)\alpha_8 \in \mathbb{F}_{2^8}$

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Fix a constant α. A *constant multiplication* is the computation of αb given b as input.

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- 14.83 bit operations on average for $\alpha \in \mathbb{F}_{2^8} \backslash \mathbb{F}_2$
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Using a linear map circuit generator

An invertible linear map circuit generator

$$\begin{array}{r|cccc}
 & (x_0, & x_1, & x_2, & x_3) \\
\phi_{\alpha_2+\alpha_4} \Big\downarrow & & & & \\
 & (x_0 + x_2 + x_3, & x_0 + x_1 + x_2 + x_3, & x_1 + x_2, & x_1 + x_3)
\end{array}$$

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- Somewhat like Paar's additive greedy common-subexpression elimination algorithm (1997)
- A **2-operand** algorithm like Bernstein's "xor-largest" algorithm (2009)

Merging radix conversions

Radix conversions:

$$\begin{aligned}f =& f^{(0)}(x^2+x) + xf^{(1)}(x^2+x)\\ =& f^{(00)}(x^4+x) + (x^2+x)f^{(01)}(x^4+x)+\\ & xf^{(10)}(x^4+x) + x(x^2+x)f^{(11)}(x^4+x)\end{aligned}$$

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Doing the size-8 first and then size-4 ones gives 12 additions.

For size-16 FFT the input is $f = \sum_{i=0}^{7} f_i x^i$:

$$\begin{aligned} f^{(00)} &= f_0 + (f_4 + f_7)x, \\ f^{(01)} &= (f_2 + f_3 + f_5 + f_6) + (f_6 + f_7)x, \\ f^{(10)} &= (f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7) + (f_5 + f_6 + f_7)x, \\ f^{(11)} &= (f_3 + f_6) + f_7 x. \end{aligned}$$

Merging radix conversions

Radix conversions:

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How about considering this as a generic linear map?

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$$

How about considering this as a generic linear map?

- 8 additions: Gao–Mateer's algorithm is suboptimal in this case

Merging radix conversions (cont.)

How about larger-size radix conversions?

Merging radix conversions (cont.)

How about larger-size radix conversions?

- Use the linear map circuit generator again

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The gain is bigger for larger sizes:

- 8 instead of 12 additions for size-8.
- 31 instead of 48 additions for size-16.
- 82 instead of 160 additions for size-32.

Merging radix conversions (cont.)

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Numbers can be further improved by an asymptotically faster radix conversion algorithm.

Faster binary-field multiplication

- bit operations for multiplication in $\mathbb{F}_{2^b}$

b	our result	competition
64	3726	≥ 3745
128	9126	≥ 11613
256	22292	≥ 34334

- competition: `http://binary.cr.yp.to/m.html`
- D'Angella et. al. (2013) have slight improvements
- Can do better for $\mathbb{F}_{2^{256}}$ with 8-way split: $f(x) = \sum_{i=0}^{7} x^i f^{(i)}(x)$