Algorithms for Model Checking (2IW55)

Lecture 2

Fairness & Basic Model Checking Algorithm for CTL and fair CTL

– based on strongly connected components –

Chapter 4.1, 4.2 + SIAM Journal of Computing 1(2), 1972

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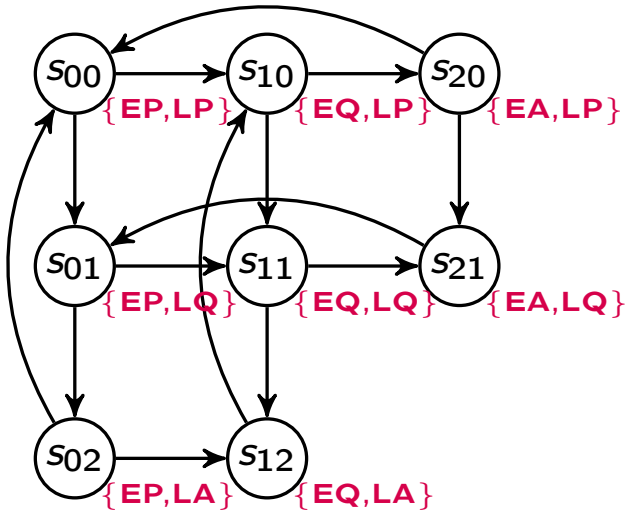
Outline

Temporal Logics: Fairness



- Atomic Propositions: EP, EQ, EA, LP, LQ, LA
- Intended meaning: Linus or Emma is either Playing, posing Questions, getting Answers
- To exclude runs in which one child gets all attention, we want that both $\neg EQ$ as well as $\neg LQ$ hold infinitely often
- fairness constraints ensuring this: $F = \{\{s_{00}, s_{01}, s_{02}, s_{20}, s_{21}\}, \{s_{00}, s_{10}, s_{20}, s_{02}, s_{12}\}\}$

Sometimes properties are violated by "unrealistic" paths only, for instance due to a scheduler. In this case, one may restrict to fair paths.

A Kripke Structure over AP with fairness constraints is a structure $M = \langle S, R, L, F \rangle$, where:

- $\langle S, R, L \rangle$ is an "ordinary" Kripke Structure as before
- $F \subseteq 2^S$ is a set of fairness constraints

A path is fair if it "hits" each fairness constraint infinitely often:

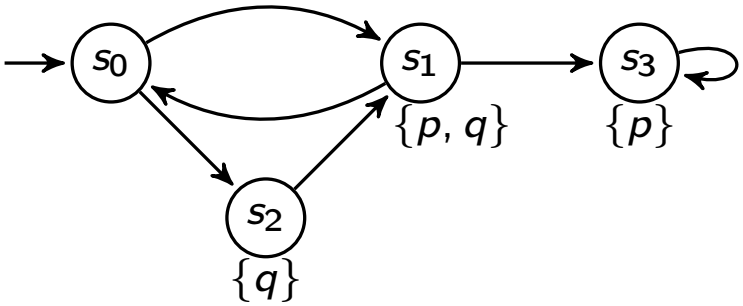
$$\text{fair}(\pi) \;\; \textit{iff} \;\; \forall C \in F.\ \{i \mid \pi(i) \in C\} \;\; \text{is an infinite set}$$

In CTL* with fairness semantics ($\models_F$), only fair paths will be considered.

Given a fixed Kripke Structure with fairness constraints $M = \langle S, R, L, F \rangle$, $s \models_F f$ means: formula f holds in state s in the fair CTL* semantics.

The definition of $\models_F$ coincides with $\models$ except for the following four clauses:

$s \models_F \text{true}$	iff	there is some fair path starting in s
$s \models_F p$	iff	$p \in L(s)$ and there is some fair path starting in s
$s \models_F \text{A } f$	iff	for all fair paths π starting in s, we have $\pi \models_F f$
$s \models_F \text{E } f$	iff	for some fair path π starting in s, we have $\pi \models_F f$



Note that $s_0 \models$ E F G p, but $s_0 \not\models$ A F G p

- First, consider as Fairness constraint: $F = \{ \{s_3\} \}$
  - then all fair paths contain s_3 infinitely often
  - we have $s_0 \models_F$ A F G p
- Next, consider as Fairness constraint: $F = \{ \{s_2\} \}$
  - then all fair paths contain s_2 infinitely often
  - in particular, fair paths cannot contain s_3
  - so $s_0 \not\models_F$ E F G p

Outline

Strongly Connected Components

Given a directed graph $G = \langle V, E \rangle$

- let $s \rightarrow_G^* t$ mean that there is a path from node s to t in G
- a strongly connected component (SCC) is a maximal subgraph S of G, such that for all $s, t \in S$, $s \rightarrow_G^* t$ and $t \rightarrow_G^* s$
- an SCC is non-trivial if it contains at least one edge

The SCCs of a graph (e.g. a Kripke Structure) can be computed in $\mathcal{O}(|V| + |E|)$ time with an algorithm based on depth-first search:

- Text book version (see Introduction to Algorithms, Corben *et al*)
- Tarjan's original algorithm (see SIAM Journal on Computing 1(2), 1972)

The second algorithm is useful in model checking contexts

Strongly Connected Components

Idea behind Tarjan's SCC algorithm
Given is a directed graph $G = \langle V, E \rangle$

- compute spanning trees by depth-first search; number the nodes in the order they are visited
- the other, non-tree edges are either:
  - forward edges (can be ignored)
  - backward edges (to an ancestor)
  - cross edges (to another subtree)

  backward and cross edges lead to nodes with smaller numbers
- nodes are kept on a stack; the nodes of a discovered SCC will be popped immediately from this stack
- compute $root[v]$: the smallest node which is:
  - reachable from v by a sequence of tree-edges followed by at most one non-tree edge; and
  - if $root[v] = v$, the root of a new SCC is found, and the whole SCC is popped from the stack

Strongly Connected Components

Procedure find_scc applies a repeated depth-first search on yet unprocessed nodes of the input graph $G = \langle V, E \rangle$
The depth-first search is delegated to the procedure dfs_scc.

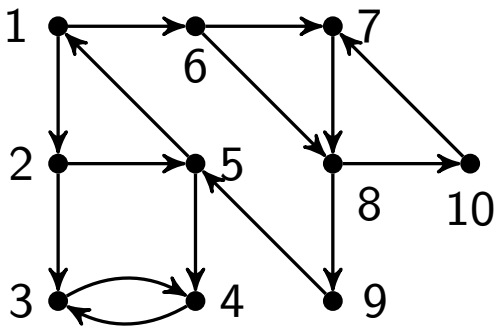
```
procedure find_scc
    i := 0;
    empty the stack;
    leave all nodes unnumbered;
    for vertices w ∈ V do
        if w is not yet numbered then
            dfs_scc(w);
        end if
    end for
end procedure
```

Strongly Connected Components

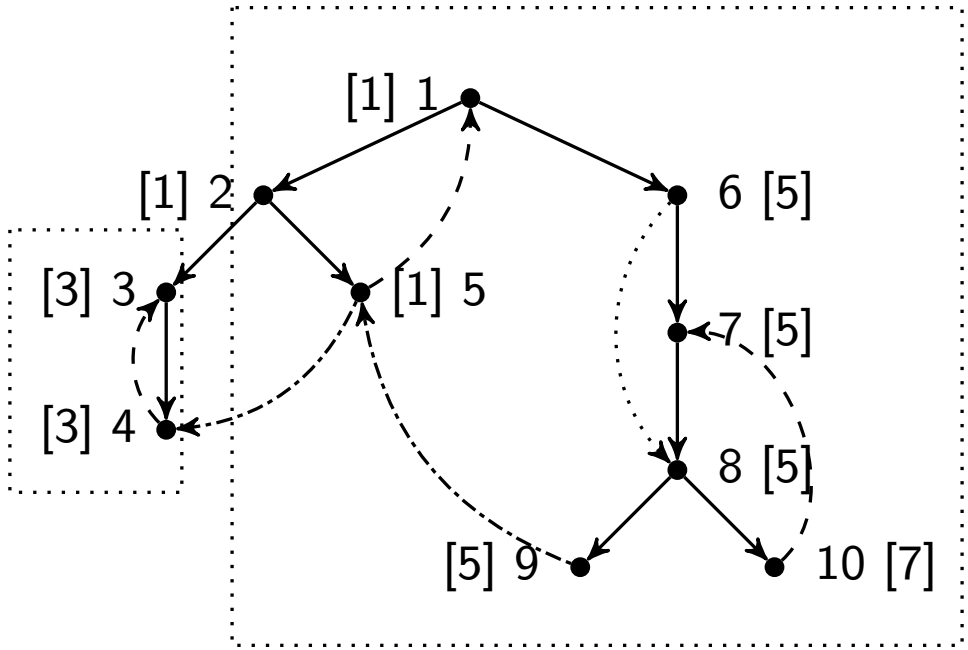

```
procedure dfs_scc(v)
    root[v] := number[v] := i := i + 1;
    push v on the stack;
    for successor w of v do
        if w is not yet numbered then                                  {tree edge}
            dfs_scc(w);
            root[v] := min(root[v], root[w]);
        else if number[w] < number[v] and w on the stack then    {cross/back edge}
            root[v] := min(root[v], number[w]);
        end if
    end for
    if root[v] = number[v] then                                    {start new SCC}
        while top w of stack satisfies number(w) ≥ number(v) do
            pop w from stack;
        end while
    end if
end procedure
```

Strongly Connected Components

Example: SCC algorithm



A possible run of the SCC algorithm, with DFS node numbers, final root-values (in square brackets), tree edges (plain arrow), forward edges (dotted), back edges (dashed), cross edges (dash/dot). Two SCCs are found: number and root value are equal



Strongly Connected Components

We analyse the space and time requirements for running find_scc on a graph $G = \langle V, E \rangle$:

- for every node:
  - dfs_scc is called exactly once
  - all its outgoing edges are explored exactly once
- each node is pushed and popped from the stack exactly once
- checking whether a node is on the stack can be done in constant time, for instance by maintaining a Boolean array

Conclusion: Tarjan's algorithm for finding strongly connected components runs in time and space $\mathcal{O}(|V| + |E|)$

Outline

CTL Model Checking Algorithm

Recall that CTL has the following ten temporal operators:

- A X and E X : for all/some next state
- A F and E F : inevitably and potentially
- A G and E G : invariantly and potentially always
- A [U] and E [U]: for all/some paths, until
- A [R] and E [R]: for all/some paths, releases

Besides atomic propositions (AP), the constant true and the Boolean connectives ($\neg, \vee$), the following temporal operators are sufficient: E X , E G , E [U].

Hence: only algorithms for computing formulae of the above form are needed.

CTL Model Checking Algorithm

Main loop of model checking CTL: check formula f on a Kripke Structure $\langle S, R, L \rangle$.

By recursion on f, algorithm mc_ctl(f) computes $label(s)$ for all states $s \in S$, where $label(s)$ shall contain those subformulae of f that hold in s.

Algorithm mc_ctl(f) employs a case distinction on the structure of f:

$f = p$	add p to $label(s)$ for those states s with $p \in L(s)$
$f = g_0 \vee g_1$	mc_ctl(g_0); mc_ctl(g_1); add f to all states labelled with g_0 or g_1
$f = \neg g$	mc_ctl(g); add f to all states not labelled with g
$f =$ E X g	mc_ctl(g); add f to all states with an R-successor labelled by g
$f =$ E [g_0 U g_1]	mc_ctl(g_0); mc_ctl(g_1); check_eu(g_0, g_1)
$f =$ E G g	mc_ctl(g); check_eg(g)

Upon termination, $s \models f$ if and only if $f \in label(s)$

CTL Model Checking Algorithm


```
procedure check_eu(f,g)
    T := {s | g ∈ label(s)};
    for all s∈T do label(s) := label(s) ∪ {E [f U g]};
    end for
    while T ≠ ∅ do
        choose s ∈ T;
        T := T \ {s};
        for all t satisfying t R s do
            if E [f U g] ∉ label(t) and f ∈ label(t)
then
                label(t) := label(t) ∪ {E [f U g]};
                T := T ∪ {t};
            end if
        end for
    end while
end procedure
```

Observations:

- label all states where g holds
- search backwards over states where f holds

CTL Model Checking Algorithm

procedure check_eg(f)
  $S' := \{s \mid f \in label(s)\}$;
  $SCC := \{C \mid C$ is a nontrivial SCC of $S'\}$;
  $T := \bigcup_{C \in SCC} \{s \mid s \in C\}$;
  for all $s \in T$ **do** $label(s) := label(s) \cup \{\text{E G } f\}$;
  end for
  while $T \neq \emptyset$ **do**
    choose $s \in T$;
    $T := T \setminus \{s\}$;
    for all t satisfying $t \in S'$ and $t\ R\ s$ **do**
      if E G $f \notin label(t)$ **then**
        $label(t) := label(t) \cup \{\text{E G } f\}$;
        $T := T \cup \{t\}$;
      end if
    end for
  end while
end procedure

Observations:

- restrict attention to subgraph where f holds
- an infinite path in a finite graph eventually reaches a non-trivial SCC

CTL Model Checking Algorithm

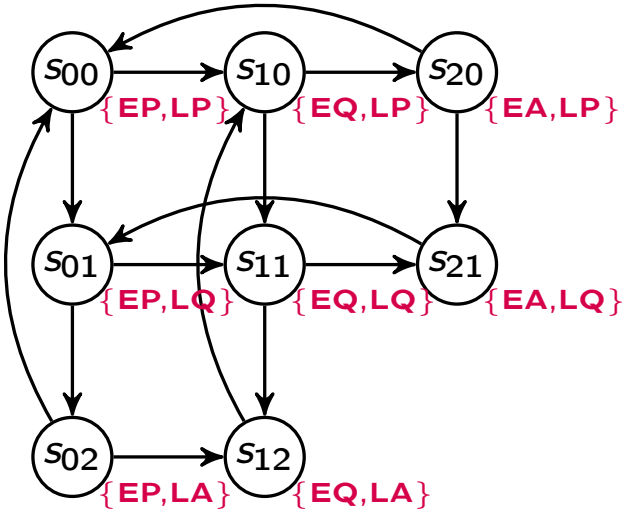
We analyse the time complexity for the standard CTL model checking algorithm of formula f (with $|f|$ the number of subformulae) on Kripke Structure $M = \langle S, R, L \rangle$.

- There are at most $|f|$ calls to mc_ctl
- Backward reachability and detecting strongly connected components can be done in time linear to the Kripke Structure: $\mathcal{O}(|S| + |R|)$
- Hence, each recursive call takes at most $\mathcal{O}(|S| + |R|)$ time

So, the complexity of this CTL model checking algorithm is $\mathcal{O}(|f| \cdot (|S| + |R|))$, which is linear in both the formula and the state space.

Outline

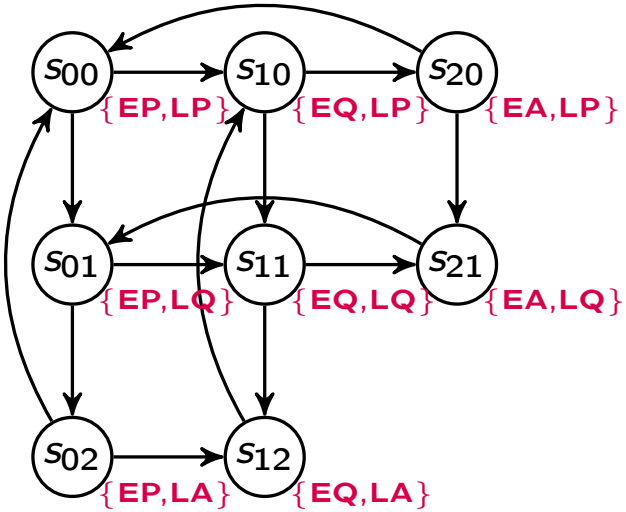
Example: demanding children



- Atomic Propositions: EP, EQ, EA, LP, LQ, LA
- Intended meaning: Linus or Emma is either Playing, posing Questions, getting Answers

Requirement: Whenever Linus asks a question, he eventually gets an answer
Formula: A G ($LQ \rightarrow$ A F LA)

Example: demanding children

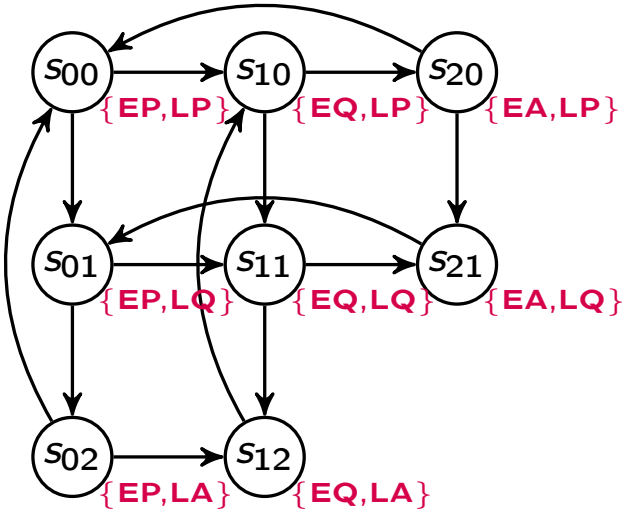


- Atomic Propositions: EP, EQ, EA, LP, LQ, LA
- Intended meaning: Linus or Emma is either Playing, posing Questions, getting Answers

- *Step 1: express using basic operators*

$$\begin{array}{l} \quad \mathsf{A\ G}\ (LQ \rightarrow \mathsf{A\ F}\ LA) \\ \equiv \\ \quad \neg \mathsf{E}\ [\mathsf{true\ U}\ \neg(\neg LQ \vee \neg \mathsf{E\ G}\ \neg LA)] \end{array}$$

Example: demanding children



- *Step 2: treat* E G $\neg LA$
  - Restrict to the subgraph where $\neg LA$ holds
  - Find non-trivial SCCs
  - Backward reachability

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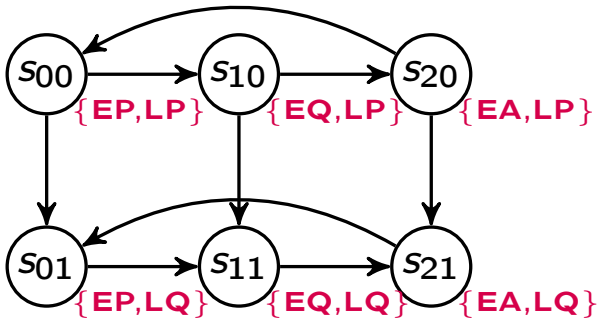

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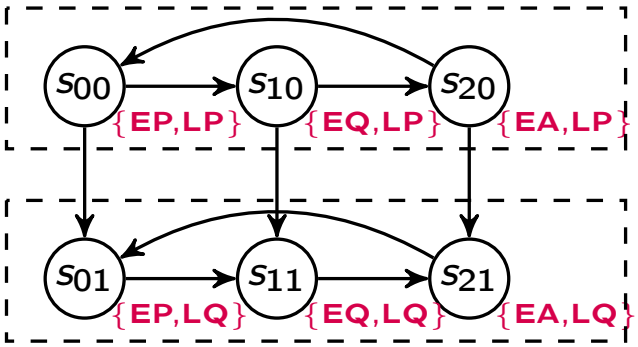

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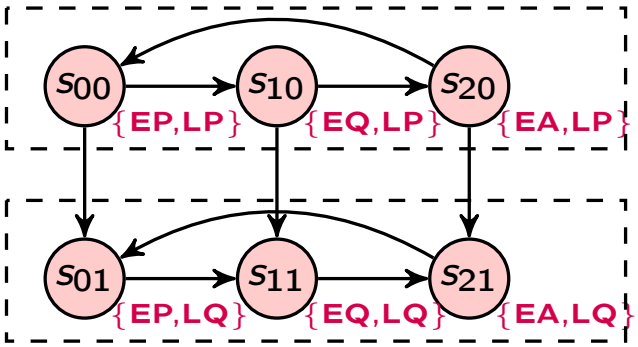

- *Step 2: treat* E G $\neg LA$
  - Restrict to the subgraph where $\neg LA$ holds
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No new states are found. So, E G $\neg LA$ holds in the states $\{s_{00}, s_{10}, s_{20}, s_{01}, s_{11}, s_{21}\}$;

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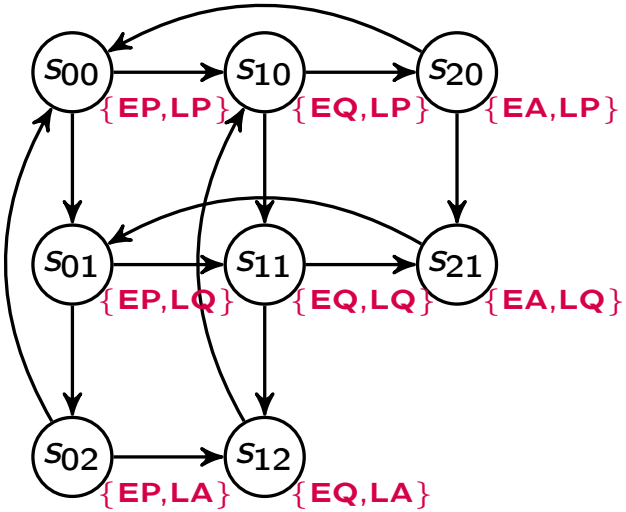

- *Step 3: treat* $\neg$E G $\neg LA$
  - E G $\neg LA$ holds in $\{s_{00}, s_{10}, s_{20}, s_{01}, s_{11}, s_{21}\}$, so $\neg$E G $\neg LA$ holds in $\{s_{02}, s_{12}\}$
- *Step 4: treat* $\neg LQ$
  - $\neg LQ$ holds in $\{s_{00}, s_{10}, s_{20}, s_{02}, s_{12}\}$
- *Step 5: treat* $\neg LQ \vee \neg$E G $\neg LA$
  - $\neg LQ \vee \neg$E G $\neg LA$ holds in $\{s_{00}, s_{10}, s_{20}, s_{02}, s_{12}\} \cup \{s_{02}, s_{12}\} = \{s_{00}, s_{10}, s_{20}, s_{02}, s_{12}\}$

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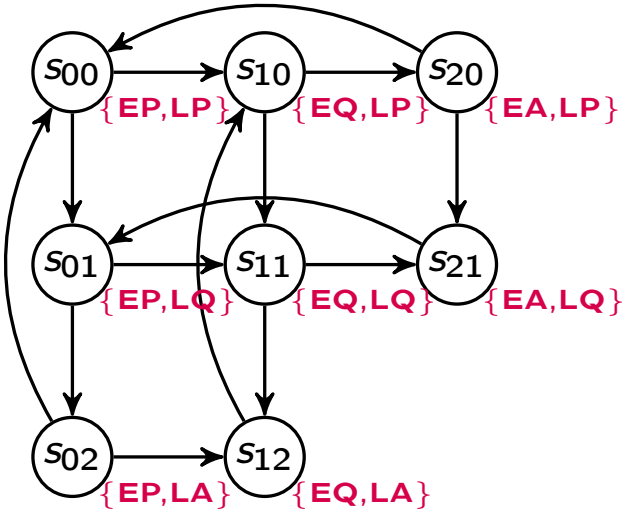

- *Step 6: treat* $\neg(\neg LQ \vee \neg \mathrm{E}\ \mathrm{G}\ \neg LA)$
  - $\neg LQ \vee \neg \mathrm{E}\ \mathrm{G}\ \neg LA$ holds in $\{s_{00}, s_{10}, s_{20}, s_{02}, s_{12}\}$, so $\neg(\neg LQ \vee \neg \mathrm{E}\ \mathrm{G}\ \neg LA)$ holds in $\{s_{01}, s_{11}, s_{21}\}$
- *Step 7: compute* $\mathrm{E}\ [\mathit{true}\ \mathrm{U}\ \neg(\neg LQ \vee \neg \mathrm{E}\ \mathrm{G}\ \neg LA)]$
  - Start in $\{s_{01}, s_{11}, s_{21}\}$
  - Perform a backward reachability analysis over states for which true holds

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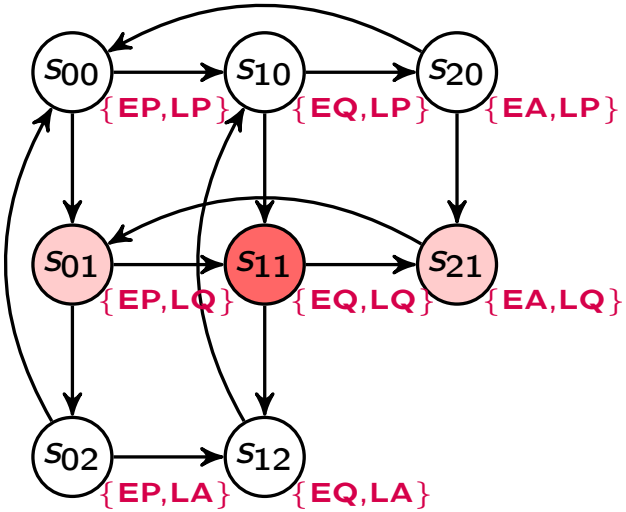

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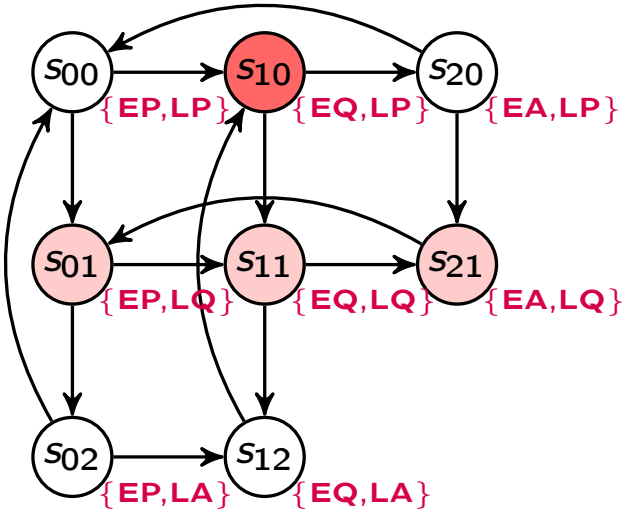

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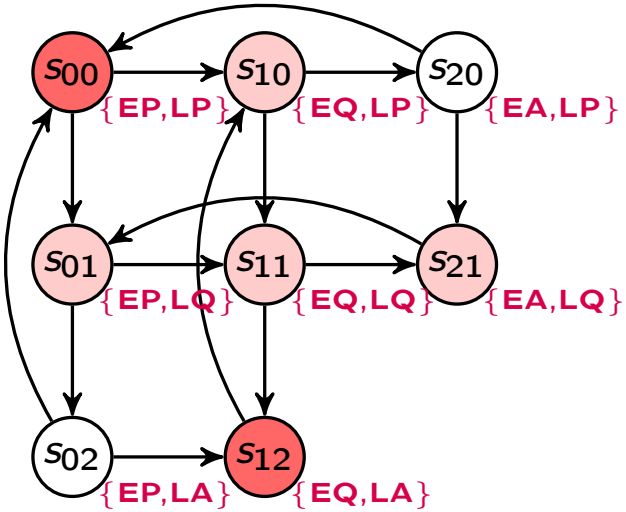

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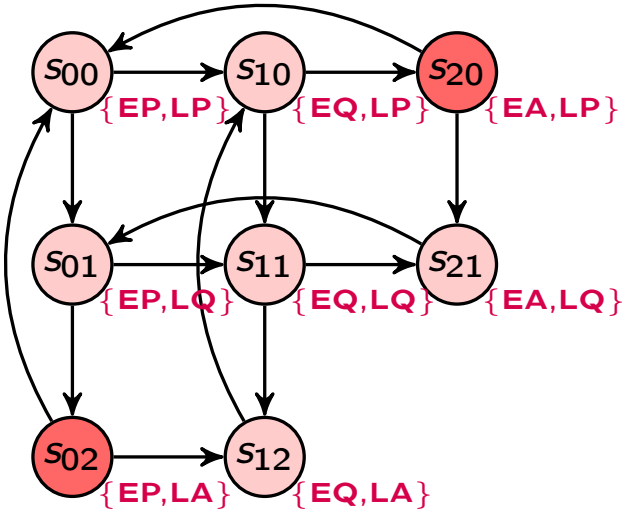

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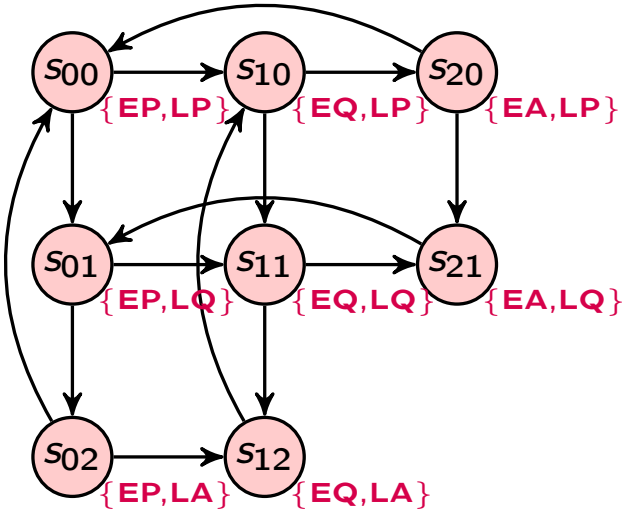

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Example: demanding children

Conclusion:

- So, E [true U $\neg(\neg LQ \vee \neg$E G $\neg LA)]$ holds in all states
- Hence, its negation A G $(LQ \rightarrow$ A F $LA)$ holds in no state
- The requirement does not hold for the full Kripke Structure
- Why? Because in this case, there is a path in which only Emma progresses while Linus is not being served.
- Next, we look at the Kripke Structure with Fairness Constraints

Outline

CTL Model Checking with Fairness

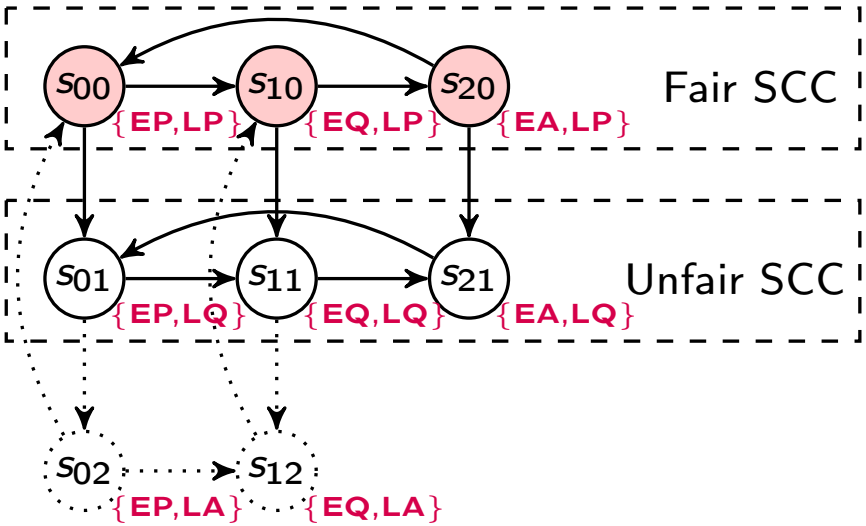
Recall: Kripke Structure $M = \langle S, R, L, F \rangle$ with fairness constraints $F \subseteq 2^S$.

- A path is fair if it "hits" each fairness constraint infinitely often
- A fair SCC is an SCC that contains an element from each constraint $C \in F$

Main idea of fair model checking for CTL:

- Special treatment for $s \models_F \mathsf{E\,G}\, f$: check_fair_eg
  - Restrict attention to $S' \subseteq S$ where f holds fairly
  - Find a path to a fair non-trivial SCC in S'
- Label states where E G true fairly holds with a new proposition symbol *fair*
- Treat the other operators using the original "unfair" procedures:
  - $s \models_F p$ ..... $s \models p \wedge \mathit{fair}$
  - $s \models_F \mathsf{E\,X}\, f$ ..... $s \models \mathsf{E\,X}\,(f \wedge \mathit{fair})$
  - $s \models_F \mathsf{E}\,[f\ \mathsf{U}\ g]$ ..... $s \models \mathsf{E}\,[f\ \mathsf{U}\ (g \wedge \mathit{fair})]$

CTL Model Checking with Fairness



- Assume fairness constraints $\neg EQ$ and $\neg LQ$.
- Remark: full graph is one big fair SCC, so E G true holds everywhere

- E G $\neg LA$:
  - Restrict to subgraph with $\neg LA$
  - Find fair non-trivial SCCs
  - Do backward reachability
- Hence: $LQ \wedge$ E G $\neg LA$ holds fairly in NO state
- Hence E F ($LQ \wedge$ E G $\neg LA$) holds nowhere fairly
- Hence, its negation, the requirement A G ($LQ \rightarrow$ A F LA) fairly holds everywhere!

Outline

Summary

CTL model checking:

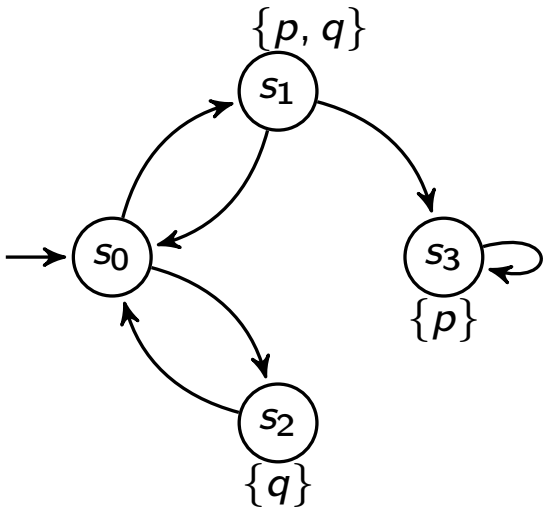
- SCC algorithm is used
- Tarjan's SCC algorithm runs one depth-first search, computing SCCs on-the-fly. Time complexity is linear
- CTL model checking can be done in time linear in the size of the formula as well as in the Kripke Structure
- Extension with Fairness Constraints is straightforward and is useful in practice
- Why not treat fairness in formulae?

$$\mathsf{A}\,[(\mathsf{G}\,\mathsf{F}\,C_1 \wedge \mathsf{G}\,\mathsf{F}\,C_2) \rightarrow \mathit{Requirement}]$$

  - fairness cannot be expressed in CTL
  - for LTL all known algorithms are exponential in the size of the formula

Outline

Exercise



CTL formulae: p, E $[q$ R $p]$, A G E F p, A G $p \vee$ A F q

- Determine for each formula in which states of the above Kripke Structure it holds; use both the semantics and use the appropriate algorithms
- Extend the Kripke structure with the Fairness constraints $F = \{ \{s_1\}, \{s_2\} \}$. In which states do the above formulae *fairly* hold?
- Similarly for the Fairness constraint $F = \{ \{s_3\} \}$