# Algorithms for Model Checking (2IW55)

## Lecture 5

## Equivalences and Pre-orders: State Space Reduction and Preservation of Properties

## Chapter 11, 11.1

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# Outline

# Equivalences



Complexity of model checking arises from:

- State space explosion: the state space is usually much larger than the specification
- Expressive logics have complex model checking algorithms

Ways to deal with the state space explosion:

- equivalence reduction: remove states with identical potentials from a state space
- on-the-fly: integrate the generation and verification phases, to prune the state space
- symbolic model checking: represent sets of states by clever data structures
- partial-order reduction: ignore some executions, because they are covered by others
- abstraction: remove details by working on conservative over-approximation

# Equivalences



- A state space reduction reduces model checking complexity.
- Of course, the reduced state space must preserve (an interesting class of) temporal properties.
- This is often characterised by an equivalence relation on Kripke Structures:
  - reduction must yield an 'equivalent" model.
  - "equivalent" models must satisfy the same properties.
- Different instances of this scheme:
  - trace equivalence preserves LTL formulae.
  - strong bisimulation preserves CTL$^*$ (and $\mu$-calculus) formulae.
  - simulation preserves ACTL$^*$ (and universal $\mu$-calculus) formulae.
  - branching bisimulation preserves CTL$^*$-X formulae.

Let two Kripke Structures over $AP$ be given:

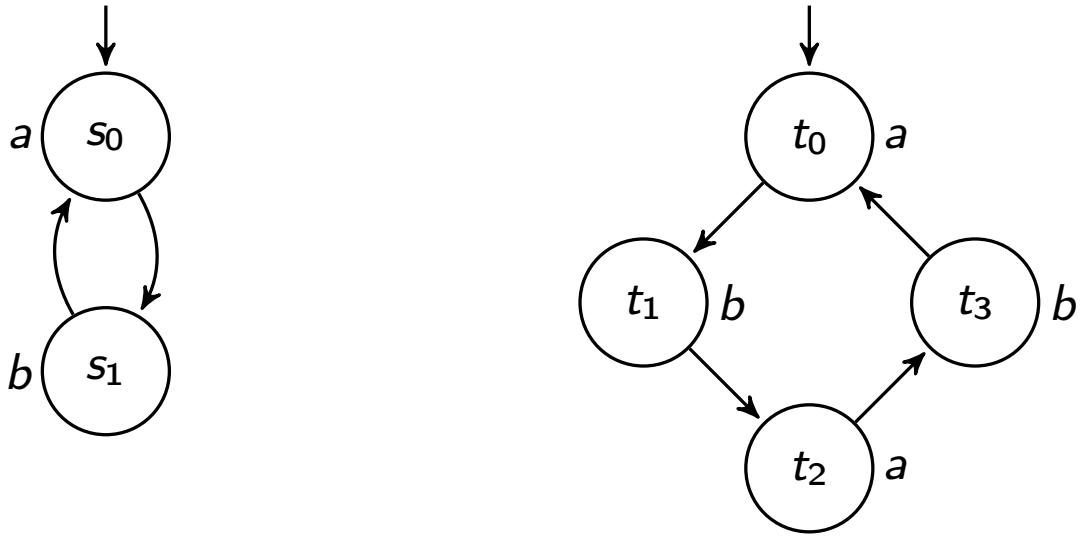
- $M = \langle S, R, S_0, L \rangle$ and
- $M' = \langle S', R', S_0', L' \rangle$

## Definition (Strong Bisimulation)

A relation $B \subseteq S \times S'$ is a strong bisimulation relation (also *zig-zag* relation) iff for every $s \in S$ and $s' \in S'$ with $sBs'$:

- $L(s) = L'(s')$
- for all $s_1 \in S$, if $sRs_1$, then there exists $s_1' \in S'$ such that $s'R's_1'$ and $s_1 B s_1'$
- for all $s_1' \in S'$, if $s'R's_1'$, then there exists $s_1 \in S$ such that $sRs_1$ and $s_1 B s_1'$

## Example



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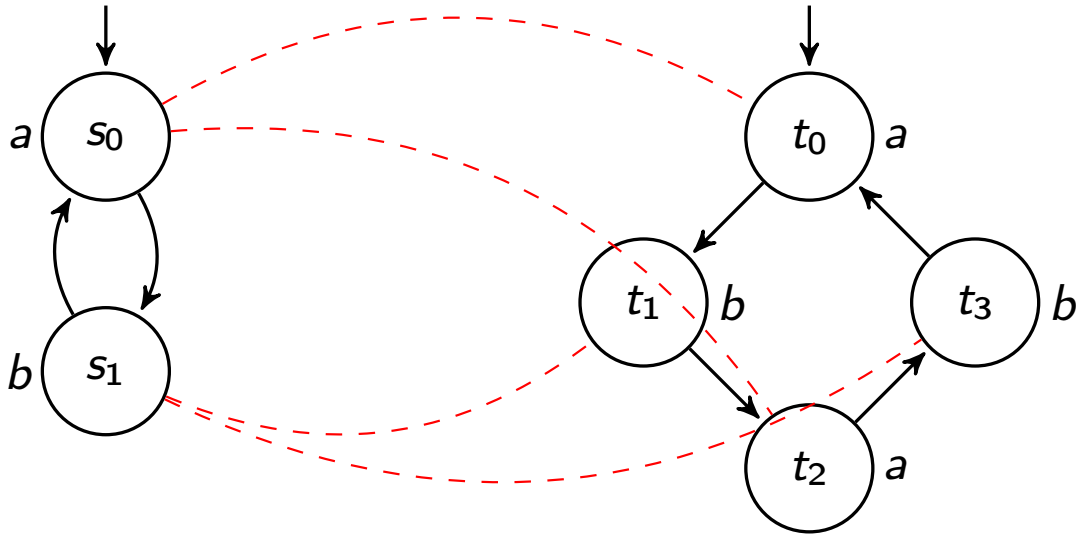


- unwinding and duplication preserves bisimulation
- Sensitive to the moment of choice

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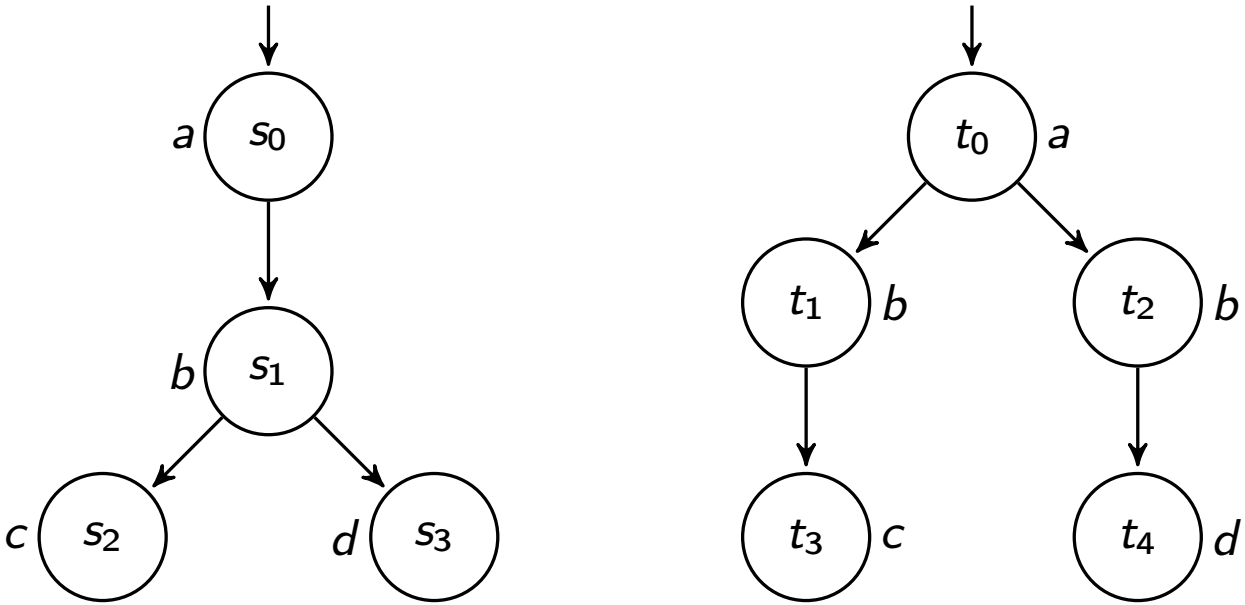

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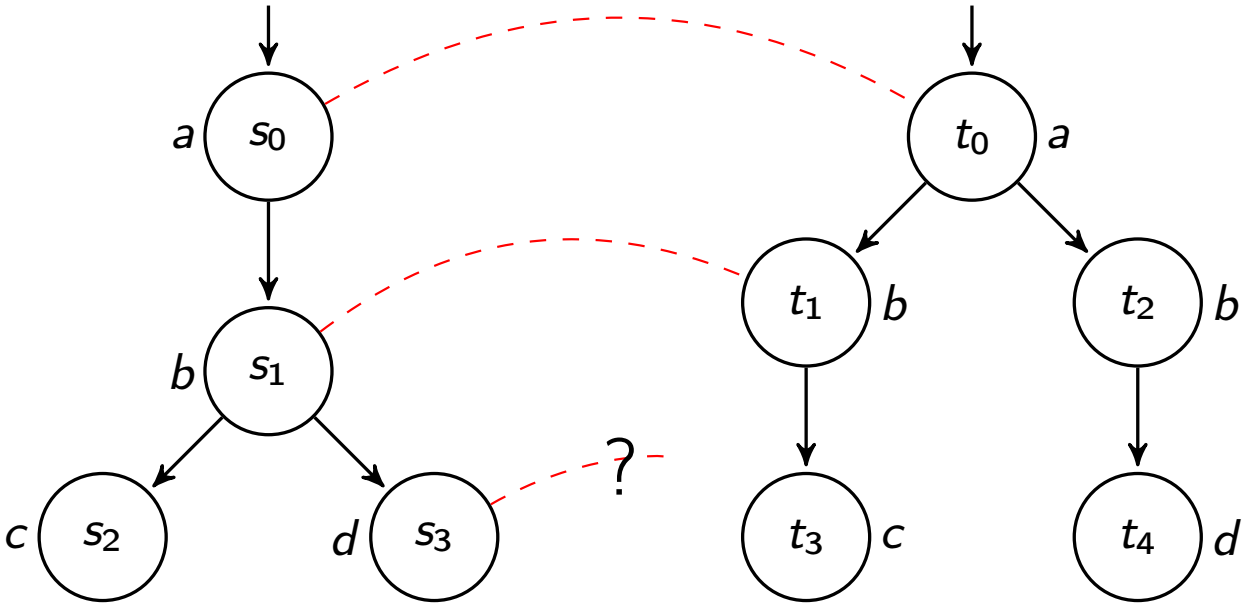


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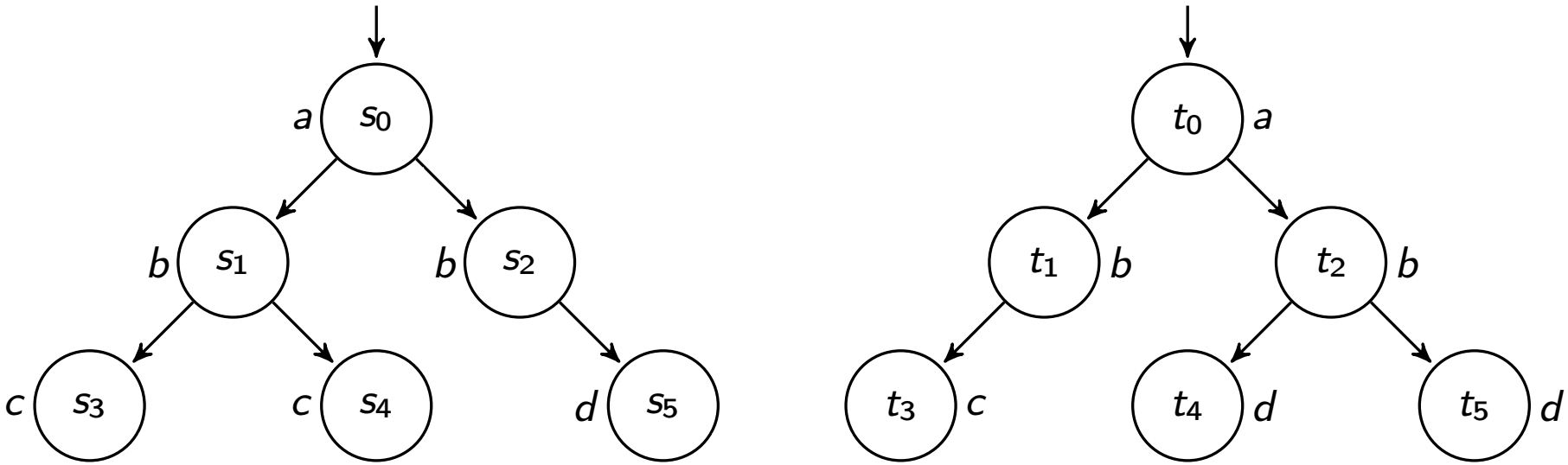
# Equivalences

## Example



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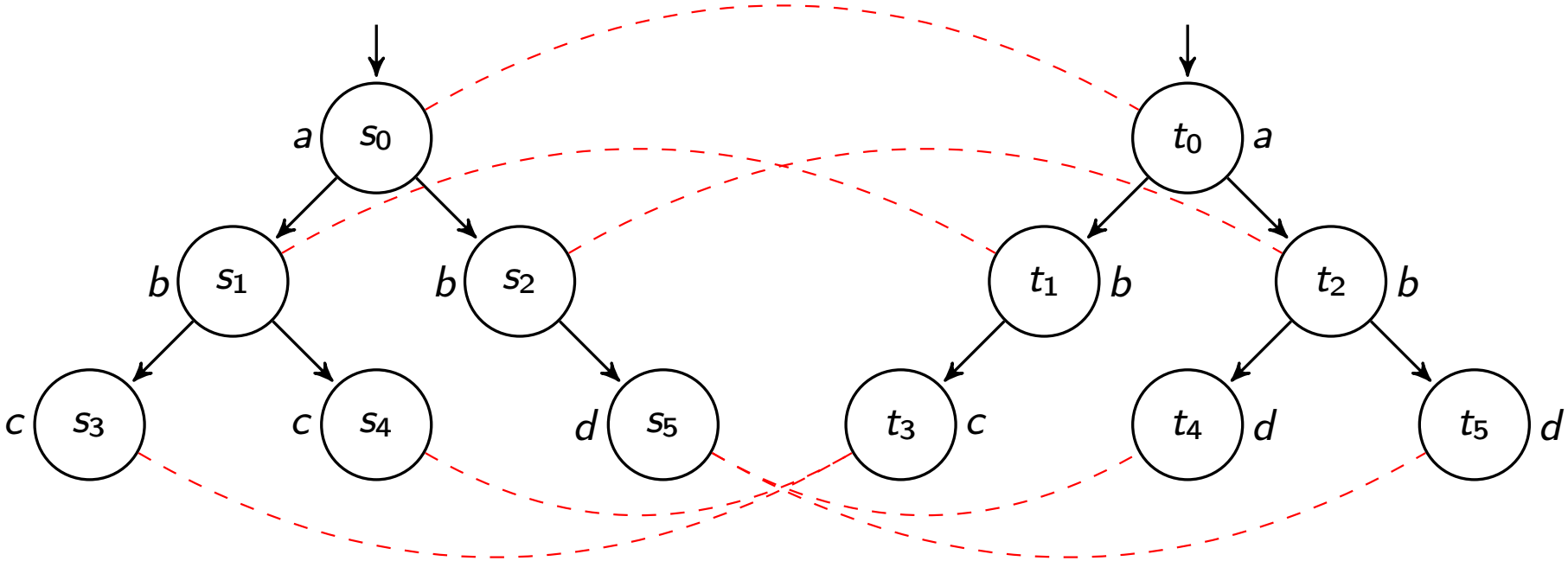



- unwinding and duplication preserves bisimulation
- Sensitive to the moment of choice

Let two Kripke Structures over $AP$ be given:

- $M = \langle S, R, S_0, L \rangle$ and
- $M' = \langle S', R', S'_0, L' \rangle$

## Definition (bisimilarity)

Two states $s \in S$ and $s' \in S'$ are bisimilar, if for some bisimulation relation $B$, $sBs'$. The Kripke Structures $M$ and $M'$ are bisimilar (notation: $M \equiv M'$) iff there exists a bisimulation relation $B$, "containing initial states", i.e.:

- $\forall s_0 \in S_0 \; \exists s'_0 \in S'_0 \; : s_0 B s'_0$
- $\forall s'_0 \in S'_0 \; \exists s_0 \in S_0 \; : s_0 B s'_0$

Note:

- bisimilarity is an equivalence relation
- the union of bisimulation relations is again a bisimulation relation
- "bisimilarity" itself is the greatest bisimulation relation

Strong bisimulation preserves CTL$^*$:

- Recall the CTL$^*$ semantics:
  - $M, s \models f$: state formula $f$ holds in state $s$,
  - $M, \pi \models f$: path formula $f$ holds along path $\pi$.
- Recall that $M \models f$ iff for all $s_0 \in S_0$, $M, s_0 \models f$.

**Theorem (14)**

*If $M \equiv M'$ (i.e. $M$ and $M'$ are bisimilar), then for every CTL$^*$ state formula $f$:*

$$M \models f \quad \textit{iff} \quad M' \models f$$

Practical consequence: In order to check $M \models f$, it is safe and sufficient to:

1. Reduce $M$ to $M'$ modulo bisimilarity,
2. Check whether $M' \models f$.

# Equivalences

Proof sketch:

Given a relation $B$, we define that path $\pi$ corresponds to path $\pi'$ iff: $\forall i.\ \pi(i)\ B\ \pi'(i)$

## Lemma (31)

*If $B$ is a bisimulation relation and $s\ B\ s'$ (correction to Lemma 31), then for every $\pi \in path(s)$ there exists a corresponding path $\pi' \in path(s')$ (and vice versa).*

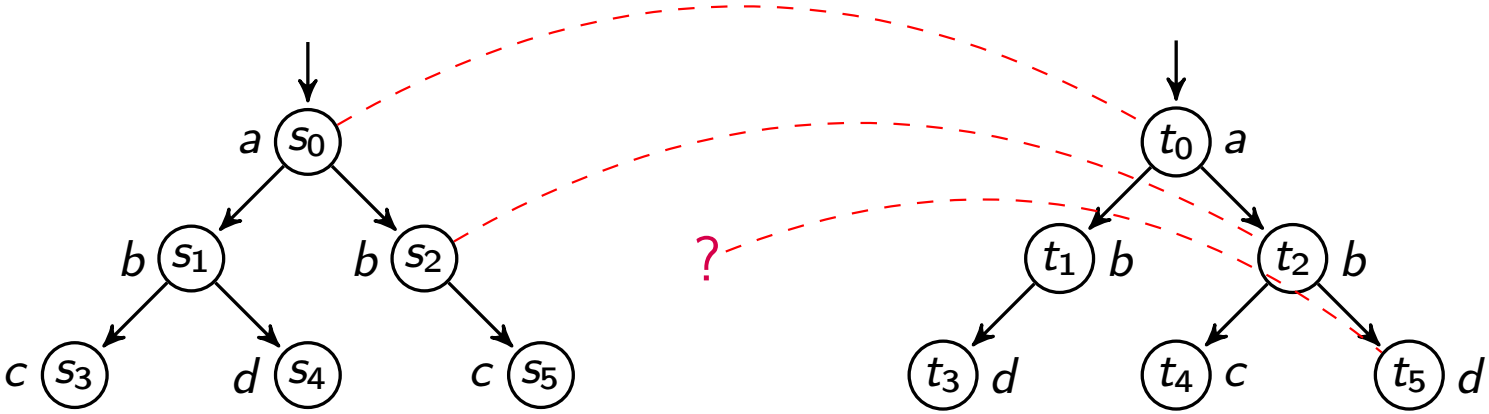
Next, with structural induction on CTL$^*$ formula $f$ one can show: if $s$ and $s'$ are bisimilar and $\pi$ and $\pi'$ correspond, then:

1. $s \models f$ if and only if $s' \models f$
2. $\pi \models f$ if and only if $\pi' \models f$

From this, the theorem follows:
for all $M, M'$ and CTL$^*$ formulae $f$: if $M \equiv M'$ then $M \models f$ iff $M' \models f$.

## Theorem (reverse)

*If $M \not\equiv M'$ then there exists a formula $f$ in CTL , such that $M \models f$ and $M' \not\models f$.*




- Note that both systems have the same paths.
- There is no bisimulation relation between these two systems containing the initial states.
- Indeed, the following CTL formula holds in (the initial state of) the right system, but not on the left: $\mathsf{A}\,\mathsf{X}\,(b \wedge \mathsf{E}\,\mathsf{X}\,d)$
- We will see later that using E is essential.

# Outline

# Pre-orders



- bisimilar models have the same behaviour, so they make true exactly the same properties.
- Idea: If we allow to really forget information, we may:
  - reduce the state space further, but:
  - preserve only a smaller class of formulae.
- We say that system $M'$ simulates system $M$ if $M'$ has at least the behaviour of $M$.

Let two Kripke Structures be given:

- $M = \langle AP, S, R, S_0, L \rangle$ and
- $M' = \langle AP', S', R', S_0', L' \rangle$, with $AP' \subseteq AP$.

## Definition (Simulation Relation)

A relation $H \subseteq S \times S'$ is a simulation relation iff for every $s \in S$ and $s' \in S'$ with $s \; H \; s'$:

- $L(s) \cap AP' = L'(s')$
- for all $s_1$, if $s \; R \; s_1$, then there exists $s_1'$ such that $s' R' s_1'$ and $s_1 \; H \; s_1'$.

## Definition (Simulation)

$M'$ simulates $M$ (written: $M \sqsubseteq M'$) iff there exists a simulation relation $H$, such that

$$\forall s_0 \in S_0.\ \exists s_0' \in S_0'.\ s_0\ H\ s_0'$$

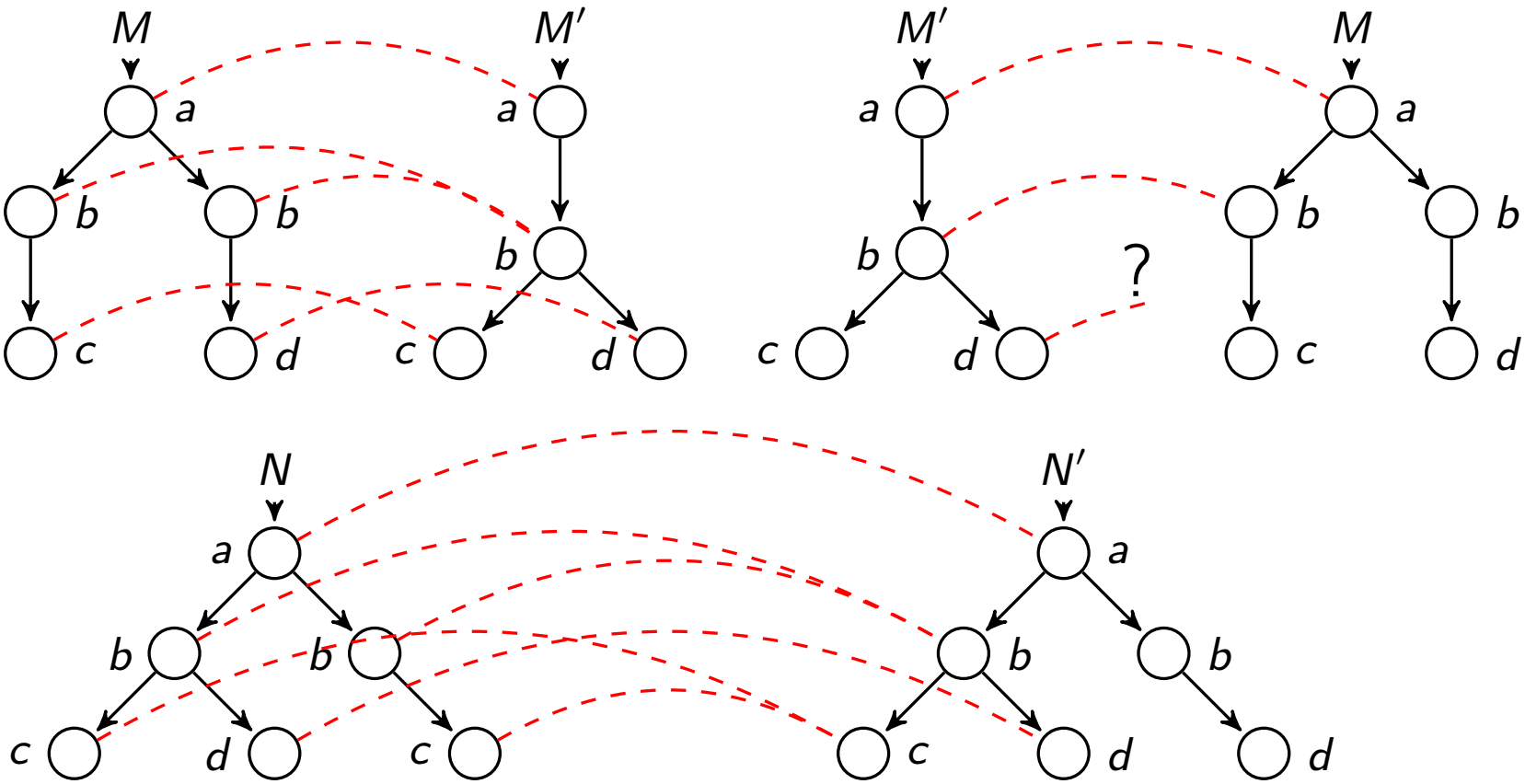
This defines an equivalence relation as follows: $M \sim M'$ iff $M \sqsubseteq M'$ and $M' \sqsubseteq M$.

Note:

- $\sqsubseteq$ is a pre-order on Kripke Structures (i.e. it is reflexive and transitive, but not necessarily symmetric).
- Warning:
  - it is possible that $M \sim M'$ but still $M \not\equiv M'$
  - In words: if two systems simulate each other, they need not be bisimilar.
  - Intuitively: the two simulations may use a different $H$, while a bisimulation requires one $B$.

# Pre-orders






- $M \sqsubseteq M'$ but not $M' \sqsubseteq M$;
- $N \sim N'$ but $N \not\equiv N'$.

## Definition (ACTL$^*$)

ACTL$^*$ (see p.31) is the fragment of CTL$^*$ with only universal path quantifiers, no existential path quantifiers.

Note:

- This only makes sense for formulae in positive normal form, i.e. negations only occur directly in front of atomic propositions.
- Examples: A F $Gp$, A G $(p \rightarrow$ A X $q)$ are in ACTL$^*$, but A G $(p \rightarrow$ E X $q)$ is not. Careful: (A G $p$) $\rightarrow$ (A G $q$) is not in ACTL$^*$, because actually:

$$\begin{aligned}(\mathsf{A}\ \mathsf{G}\ p) \rightarrow (\mathsf{A}\ \mathsf{G}\ q) &\equiv \neg(\mathsf{A}\ \mathsf{G}\ p) \vee (\mathsf{A}\ \mathsf{G}\ q) \\ &\equiv (\mathsf{E}\ \mathsf{F}\ \neg p) \vee (\mathsf{A}\ \mathsf{G}\ q)\end{aligned}$$

# Pre-orders



Simulation preserves ACTL$^*$:

**Theorem**
*If $M \sqsubseteq M'$ (i.e. $M'$ simulates $M$), then for every ACTL$^*$ state formula $f$ over AP':*

$$\text{if } M' \models f \quad \text{then} \quad M \models f$$

Practical consequence: In order to check $M \models f$, it is safe to find an approximation $M'$ with $M \sqsubseteq M'$ and check that $M' \models f$.

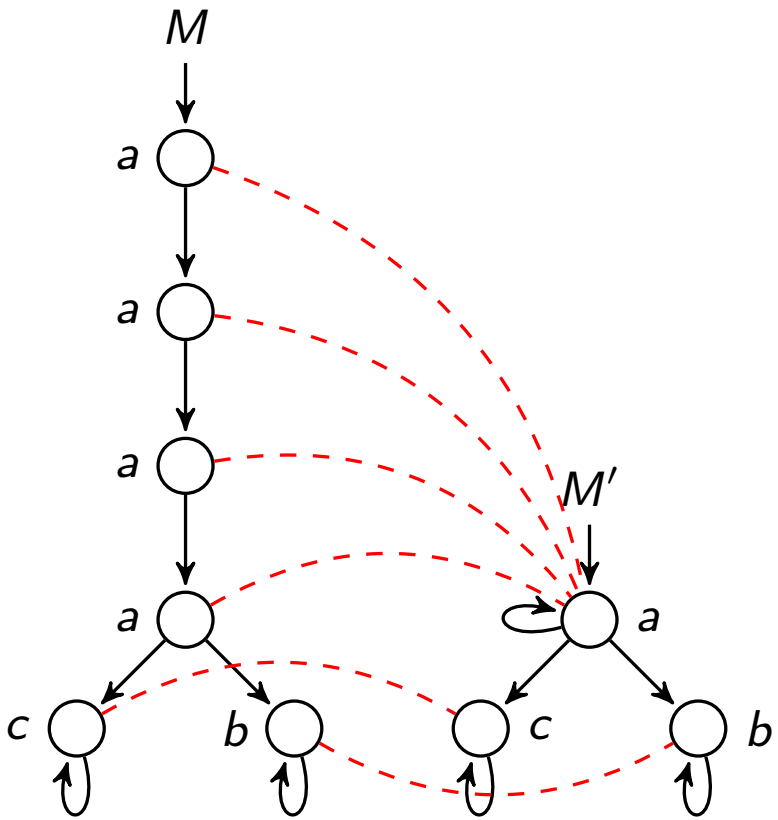
However: if $M' \not\models f$, we obtain no information about $M \models f$ — it may or may not hold.

In the previous example, we had: $N \sim N'$ but $N \not\equiv N'$. Hence:

- $N$ and $N'$ satisfy the same ACTL$^*$ formulae
- $N$ and $N'$ do not satisfy the same CTL formulae
- They can only be distinguished using operator E .

# Pre-orders

## Example




- Observe that $M \sqsubseteq M'$ with $H$ indicated left.
- Note that $M' \models \mathsf{A}\,\mathsf{G}\,(a \vee b \vee c)$ and hence $M \models \mathsf{A}\,\mathsf{G}\,(a \vee b \vee c)$.
- Note that $M' \not\models \mathsf{A}\,\mathsf{F}\,(b \vee c)$, but actually $M \models \mathsf{A}\,\mathsf{F}\,(b \vee c)$. This shows that some information is really lost.
- Note: $M \models \mathsf{A}\,\mathsf{X}\,a$ but $M' \not\models \mathsf{A}\,\mathsf{X}\,a$. Conclusion: $M' \not\sqsubseteq M$.
- Note: $M' \models \mathsf{E}\,\mathsf{X}\,b$, but $M \not\models \mathsf{E}\,\mathsf{X}\,b$ (not in ACTL$^*$, is in ECTL$^*$).

# Outline

# Bisimulation Reduction



Computing Bisimulation Equivalence:

Let two Kripke Structures be given:

- $M = \langle AP, S, R, S_0, L \rangle$ and
- $M' = \langle AP, S', R, S'_0, L' \rangle$.

Define a sequence of relations $s\ B_i^*\ s'$ iff $s$ and $s'$ cannot be distinguished within $i$ steps:

- $s\ B_0^*\ s'$ if and only if $L(s) = L'(s)$.
- $s\ B_{n+1}^* s'$ if and only if:
  1. $s\ B_n^*\ s'$, and
  2. $\forall s_1$ with $R(s, s_1)$, $\exists s'_1$ with $s'\ R'\ s'_1$ and $s_1\ B_n^*\ s'_1$.
  3. $\forall s'_1$ with $R'(s', s'_1)$, $\exists s_1$ with $s\ R\ s_1$ and $s_1\ B_n^*\ s'_1$.
- Let $B^* := \bigcap_i B_i^*$

Clearly, $B_i^* \supseteq B_{i+1}^*$, so $B^*$ can be computed by fixed point iteration.

Actually, this can be implemented symbolically by OBDDs

# Bisimulation Reduction



- Observe: $B^*$ is the largest bisimulation between $M$ and $M'$.
- So: if $s$ and $s'$ are bisimilar, then $s\ B^*\ s'$.
- To test if $M \equiv M'$: check if for each $s_0 \in S_0$ there exists an $s_0' \in S_0'$ such that $s_0\ B^*\ s_0'$.
- By carefully splitting equivalence classes, the procedure can run in $\mathcal{O}(|R| \times \log(|S|))$ time (Paige-Tarjan).
- Similar ideas apply to checking $M \sqsubseteq M'$.

The algorithm can be modified for state space reduction as follows:

- The equivalence classes of $B^*$ form the states of the reduced state space (minimal modulo bisimulation).
- The transitions between two classes are derived from the transitions between elements of these classes.

# Outline

# Summarising



- Bisimulation is an equivalence relation.
- Bisimulation preserves CTL$^*$ formulae.
- Simulation is a pre-order.
- Simulation preserves ACTL$^*$ formulae only, and only in one direction.
- Simulation allows for more reduction but sometimes crucial information is lost.
- Bisimulation and Simulation reduction can be computed in polynomial time.

Possible improvement: Instead of:

1. generating state space
2. reducing state space
3. model checking reduced state space,

it would be better to generate a smaller state space immediately.