Exercises

29 March, 2017, 10:45 – 12:30

1. This question is composed of four subquestions.

   (a) Give a parity game $G = (V, E, p, (V_\Diamond, V_\Box))$, consisting of 6 vertices and ensure that G is such that:

      i. for every $1 \leq i \leq 6$, there is a vertex v such that $p(v) = i$; *i.e.*, all priorities are different and taken from the range $1 \ldots 6$,

      ii. G has at least 10 edges,

      iii. player $\Diamond$ **has no** strategy σ that guarantees that priority 6 occurs infinitely often on **all** plays (i.e. plays starting in arbitrary vertices) consistent with σ,

      iv. player $\Box$ **has** a memoryless strategy ρ such that for **all** plays π (starting in arbitrary vertices) consistent with ρ, at some point 2 vertices with **odd** priority appear in π, *without* any vertex with an **even** priority in between these two vertices with a priority lower than those two odd priorities,

      v. all vertices in G are won by player $\Diamond$.

   (b) Prove that your game G of question (a) satisfies property (iii).

   (c) Define a memoryless strategy ρ for player $\Box$ and show that the game G of question (a) satisfies property (iv).

   (d) Prove using either Zielonka's recursive algorithm or by transforming game G of question (a) to a BES and solving the resulting BES, that your solution meets property (v). In case you use the recursive algorithm, clearly indicate which subgames are solved in each recursive step. In case of converting the problem to Boolean equation systems, clearly indicate how the solution to the equation system is linked to the solution of the parity game.

2. Consider the following parameterised Boolean equation system

$$
\begin{aligned}
&(\nu X(n : Nat) && = \forall k : Nat.\ Y(n,k)) \\
&(\nu Y(n : Nat, k : Nat) && = ((n = k) \wedge Z(n,k)) \vee ((n \neq k) \wedge X(k))) \\
&(\mu Z(n : Nat, k : Nat) && = Y(n+1, k+1))
\end{aligned}
$$

   (a) Compute the solution to X in the above parameterised Boolean equation system using symbolic approximation and Gauß Elimination. Include the important steps in the computations in your answer.

   (b) In case $X(0)$ has solution false, give a refutation graph explaining this, and, in case $X(0)$ has solution true, give a proof graph explaining this. Formally state the set of vertices, edges and labelling of the graph.

3. Let N be the natural numbers, with addition $(+)$ and modulo (mod) and let B be the Booleans. Let $\mathcal{E}$ be the parameterised Boolean equation system depicted below:

$$
\begin{aligned}
&\nu X(n : N) && = Y(n+1, n \mathbin{\mathsf{mod}} 2 = 0) \\
&\mu Y(n : N, b : B) && = (b \vee Z(n+1)) \wedge (\neg b \vee Z(n+2)) \wedge X(n+1) \\
&\nu Z(n : N) && = Z(n+1) \wedge Z(n+2)
\end{aligned}
$$

   Compute the solution to $X(0)$, where X is defined by $\mathcal{E}$, using instantiation to a Boolean equation system and solving the latter through both Gauß Elimination and by converting the Boolean equation system to a parity game and solving the latter. Clearly indicate all steps and transformations you use in your computation.

$\Box$