Algorithms for Model Checking (2IW55)

Lecture 11

Parity games

Background material:

Chapter 3 of "An experimental study of algorithms and optimisations for parity games, with an application to Boolean Equation Systems",

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HG 6.81

Parity games

Today: verification problem represented by parity games:

- graph game
- total graph
- two players: ◇ and □ (Even and Odd)
- vertices:
  - integer priority $p(v)$
  - owned by one player

Playing the game:

- token on vertex
- player does step to successor vertex if token is on vertex owned by him
- play: infinite sequence of steps

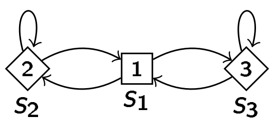
Parity games (definition)

Definition (Parity game)

A parity game Γ is a four tuple $(V, E, p, (V_\diamond, V_\Box))$ where

- (V, E) is a directed graph
- V a set of vertices
- E a total edge relation
- $\mathrm{p} : V \to \mathbb{N}$ a priority function
- $(V_\diamond, V_\Box)$ a partitioning of V, i.e. $V = V_\diamond \cup V_\Box$ and $V_\diamond \cap V_\Box = \emptyset$

Parity game (example)



$$
\begin{array}{lcl}
V_\diamond & = & \{s_2, s_3\} \\
V_\Box & = & \{s_1\} \\
\mathrm{p} & = & \{s_1 \mapsto 1, s_2 \mapsto 2, s_3 \mapsto 3\}
\end{array}
$$

Motivation

- BES can blow up exponentially (see e.g. Mader, section 6.4.2)
- Semantics of BES hard to understand
- Alternative model:
  - additional insights
  - other algorithms
  - graph model more intuitive and easier to understand
  - strong link with BES
- Algorithms still exponential

Winning a parity game

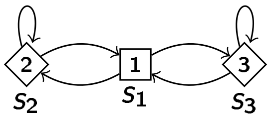
Definition (Winner)

Let:

- $\pi = v_1, v_2, v_3, \dots$ be a play
- $\inf(\pi)$ be the set of priorities occurring infinitely often in π

Play π is winning for player $\Diamond$ iff $\min(\inf(\pi))$ is even

Winning a parity game (example)



- Play $s_1 s_2^\omega$ won by player $\Diamond$;
- Play $(s_1 s_2 s_1 s_3)^\omega$ won by player $\Box$.

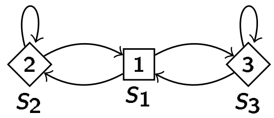
Strategies

Definition (Strategy)

A strategy for $\bigcirc$ is a partial function $\varrho_\bigcirc : V^* \times V_\bigcirc \to V$ that decides the vertex the token is played to based on the history of the vertices that has been visited.

- A play $\pi = v_1, v_2, v_3, \ldots$ is consistent with strategy $\varrho_\bigcirc$ for $\bigcirc$ iff every $v_i \in \pi$ such that $v_i \in V_\bigcirc$ is immediately followed by $v_{i+1} = \varrho_\bigcirc(v_1, \ldots, v_i)$.

Strategy (example)



- Consider strategy $\varrho_\Box$ that, from s_1 plays token to s_2 if s_1 has been visited an even number of times, and to s_3 otherwise.
- What if $\varrho_\Diamond$ always plays token from s_2 to s_2?

Winning

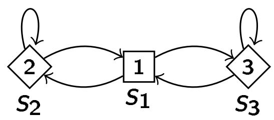
Definition (Winning strategy)

Strategy $\varrho_\bigcirc$ is a winning strategy for $\bigcirc$ from set $W \subseteq V$ if every play starting from a vertex in W, consistent with $\varrho_\bigcirc$ is winning for $\bigcirc$.

Goal of solving parity games: determine unique partitioning $(W_\Diamond, W_\Box)$ of V such that:

- player $\Diamond$ has winning strategy from $W_\Diamond$
- player $\Box$ has winning strategy from $W_\Box$

Winning strategy (example)



- $\varrho_\Diamond(\ldots, s_2) = s_2$
- $\varrho_\Box(\ldots, s_1) = s_3$
- $\varrho_\Diamond(\ldots, s_3) = \begin{cases} s_1 & \text{if number of occurrences of } s_3 \text{ is prime} \\ s_3 & \text{otherwise} \end{cases}$

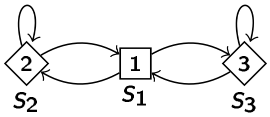
Memoryless strategies

Theorem
For finding winning strategies it suffices to look at history free (also memoryless) strategies.

History free strategy is function $\varrho_\bigcirc : V_\bigcirc \to V$, such that:

- vertex v_i always gets the same successor v_{i+1}
- independent of path by which v_i is reached.

Memoryless strategy (example)



Let $\varrho_\Diamond(s_2) = s_2$, $\varrho_\Diamond(s_3) = s_1$, and $\varrho_\Box(s_1) = s_3$.

- $\varrho_\Diamond$ is winning from $\{s_2\}$
- $\varrho_\Box$ is winning from $\{s_1, s_3\}$

Boolean Equation Systems

A Boolean expression is defined by:

$$f \ ::= X \mid \mathsf{true} \mid \mathsf{false} \mid f \wedge f \mid f \vee f$$

A Boolean equation is an equation of the form

$$\sigma X = f$$

A Boolean Equation System (BES) is a sequence of Boolean quations:

$$\mathcal{E} ::= \varepsilon \mid (\sigma X = f)\mathcal{E}$$

BES (example)

$$\begin{array}{lcl} \mu X & = & X \wedge (Y \vee Z) \\ \nu Y & = & W \vee (X \wedge Y) \\ \mu Z & = & \text{false} \\ \mu W & = & Z \vee (Z \vee W) \end{array}$$

Notation

Rank:

- natural number
- lowest rank alway 0 or 1
- rank(X) indicates in which block of like-signed equations X occurs
- rank(X) is odd iff X is defined in a μ-equation
- rank(X) inductively defined on structure of BES

Operand:

- $\wedge$, $\vee$ or $\perp$
- op(X) indicates top-level boolean operator of equation for X

Rank/Op (example)

rank(_)	op(_)			
(1)	$\wedge$	μX	$=$	$X \wedge (Y \vee Z)$
(2)	$\vee$	νY	$=$	$W \vee (X \wedge Y)$
(3)	$\perp$	μZ	$=$	false
(3)	$\vee$	μW	$=$	$Z \vee (Z \vee W)$

Standard Recursive Form

A BES is in Standard Recursive Form (SRF) if all right hand sides of Boolean Equations adhere to the following syntax:

$$f := X \mid \bigvee F \mid \bigwedge F$$

- X is a proposition variable
- F is a non-empty set of proposition variables

Observe that:

- all BESs can be transformed into an equivalent BES in SRF
- this transformation can be done in polynomial time

Transformation to SRF

- Introduce equations for subformulae with different Boolean operand in last block of BES
- Introduce equation $\mu Xf = Xf$ for false, $\nu Xt = Xt$ for true, in arbitrary existing block if possible; replace all occurrences of false with Xf, and true with Xt

Consider the following BES:

$$\begin{array}{lcl} \mu X & = & X \wedge (Y \vee Z) \\ \nu Y & = & W \vee (X \wedge Y) \\ \mu Z & = & \text{false} \\ \mu W & = & Z \vee (Z \vee W) \end{array}$$

This corresponds to the following BES in SRF:

$$\begin{array}{lcl} \mu X & = & X \wedge X' \\ \nu Y & = & W \vee Y' \\ \mu Z & = & Xf \\ \mu W & = & Z \vee (Z \vee W) \\ \mu Xf & = & Xf \\ \nu X' & = & Y \vee Z \\ \nu Y' & = & X \wedge Y \end{array}$$

Parity games vs BES

Definition (Parity game to BES)

Let $(V, E, p, (V_\Diamond, V_\Box))$ be a parity game.
The corresponding closed BES in SRF contains the following equations:

$$\begin{cases} \sigma X_v = \bigwedge\{X_{v'} \mid (v, v') \in E\} & \text{if } v \in V_\Box \\ \sigma X_v = \bigvee\{X_{v'} \mid (v, v') \in E\} & \text{if } v \in V_\Diamond \end{cases}$$

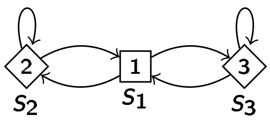
where:

- $\sigma = \mu$ if $p(v)$ is odd, $\sigma = \nu$ otherwise
- X_v occurs before X_u in the BES if $p(v) < p(u)$

Theorem

Solution to X_v is true $\Leftrightarrow$ player $\Diamond$ has winning strategy from v

Parity game to BES example



Corresponds to the following BES:

$$\begin{aligned}
\mu X_{s_1} &= X_{s_2} \wedge X_{s_3} \\
\nu X_{s_2} &= X_{s_2} \vee X_{s_1} \\
\mu X_{s_3} &= X_{s_1} \vee X_{s_3}
\end{aligned}$$

Dependency graph of a BES in SRF

Let $\mathcal{E}$ be a BES, then

- bnd($\mathcal{E}$): variables occurring at the lhs of an equation in $\mathcal{E}$;
- occ($\mathcal{E}$) variables occurring at the rhs of an equation in $\mathcal{E}$;
- occ(f) are all variables occurring in formula f.

Definition (Dependency graph)

Let $\mathcal{E}$ be a BES. Its dependency graph $\mathcal{G}_{\mathcal{E}}$ is defined as (V, E), where:

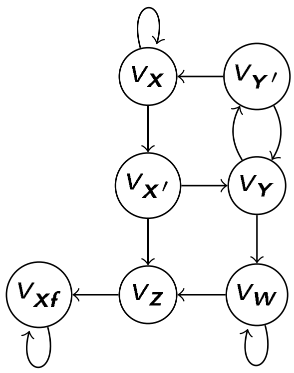
- $V = \{V_X \mid X \in \mathrm{bnd}(\mathcal{E})\}$
- $(V_X, V_Y) \in E$ iff there is $\sigma X = f$ in $\mathcal{E}$ with $Y \in \mathrm{occ(f)}$

Dependency graph (example)

Consider the following BES:

$$\begin{array}{lcl}
\mu X & = & X \wedge X' \\
\nu Y & = & W \vee Y' \\
\mu Z & = & Xf \\
\mu W & = & Z \vee (Z \vee W) \\
\mu Xf & = & Xf \\
\nu X' & = & Y \vee Z \\
\nu Y' & = & X \wedge Y
\end{array}$$

Its dependency graph is:



BES vs parity game

Definition (BES to parity game)

Let $\mathcal{E}$ be a closed BES in SRF. This corresponds to the parity game $\Gamma = (V, E, p, (V_\diamond, V_\Box))$, where

- (V, E) is the dependency graph $\mathcal{G}_\mathcal{E}$ of $\mathcal{E}$,
- $p(V_X) = \text{rank(X)}$ for all variables $X \in \text{bnd(E)}$,
- $V_\Box = \{V_X \mid \text{op(X)} = \wedge\}$, so all conjunctive equations are assigned to $V_\Box$, and
- $V_\diamond = V \setminus V_\Box$, all other equations are assigned to $V_\diamond$.

Theorem

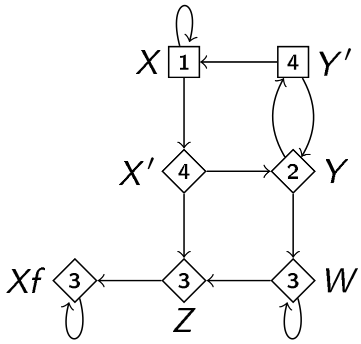
Player $\diamond$ has winning strategy from V_X $\Leftrightarrow$ the solution of X is true

BES vs parity game (example)

Consider the following BES:

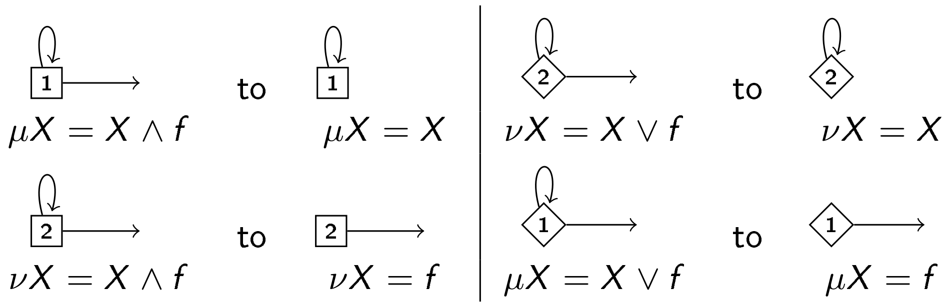
$$
\begin{array}{lcl}
\mu X & = & X \wedge X' \\
\nu Y & = & W \vee Y' \\
\mu Z & = & Xf \\
\mu W & = & Z \vee (Z \vee W) \\
\mu Xf & = & Xf \\
\nu X' & = & Y \vee Z \\
\nu Y' & = & X \wedge Y
\end{array}
$$

Its parity game is:



Transformations on parity games

Self-loop elimination

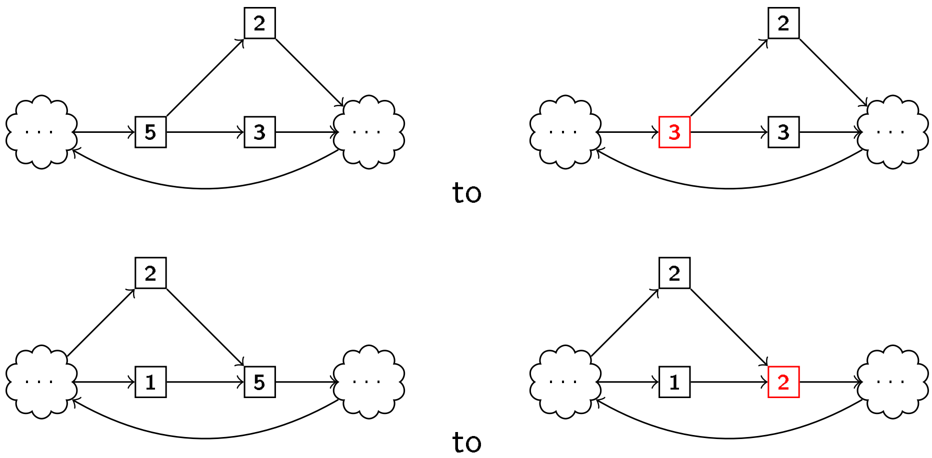


Priority compaction

In case priority 4 does not occur in the parity game. Evenness must be preserved!

Transformations on parity games

Priority propagation



Corresponds to re-ordering of equations in BES, which is generally unsafe!

Bisimulation

Definition (Bisimilarity of vertices)

Let $G = (V, E, \mathrm{p}, (V_\Diamond, V_\Box))$ be a parity game. Let R be a symmetric relation. R is a bisimulation relation if $v \mathrel{R} v'$ implies

- $v \in V_\Diamond \Leftrightarrow v' \in V_\Diamond$
- $\mathrm{p}(v) = \mathrm{p}(v')$
- $v \to w$ implies $\exists w'$ such that $v' \to w'$ and $w \mathrel{R} w'$

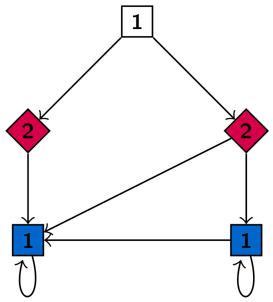
Vertices v and v' are bisimilar ($v \equiv v'$) iff there exists a bisimulation relation R such that $v \mathrel{R} v'$.

Theorem

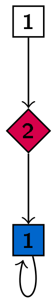
$v \equiv v'$ implies that v and v' are won by the same player

Bisimilarity (example)

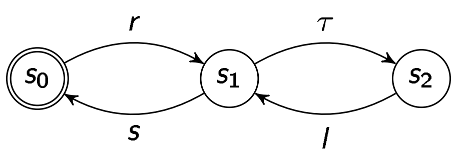
Original



Minimal bisimilar parity game



Exercise



Consider the following modal μ-calculus formula f:

$$\phi \equiv \nu X.\mu Y.(([r]X \wedge [s]X \wedge (\nu Z.\langle \neg s \rangle Z)) \vee ([r]Y \wedge [s]Y))$$

- Translate the model checking question $M \models f$ to a BES.
- Transform the resulting BES into a parity game.
- Determine the winner from vertex $V_{X_{s_0}}$.
- Provide winning strategy for this player from $V_{X_{s_0}}$.