Simulation Equivalence Reduction in Parity Games
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Abstract

The introduction of the notion of direct simulation on parity games gives rise to possibilities for efficient reduction of these structures, preserving their outcomes. We evaluate the reduction of parity games using the simulation equivalence relation as a basis. To this end, a similar notion on Kripke structures is extended to the domain of parity games and correctness is proven. The resulting structure is proved minimal and an ordering is provided to find a unique minimal parity game.

1 Introduction

We consider the field of model checking, which deals with proving properties of systems of concurrent processes. In order to avoid traversing the entire state space of the model one state at a time, a symbolic approach is used.

It is possible to describe many desired temporal properties in the modal μ calculus ([6], [2]). These properties can be combined with the model and then be translated into boolean equation systems ([7]). These systems can in turn be translated into parity games ([4]). Various algorithms exist to solve these parity games in order to prove or disprove whether these properties properties hold on the modelled system ([5]).

Because all known algorithms for solving parity games have a superpolynomial running time, it may be beneficial to reduce these structures before solving them. As a basis for this reduction, we assume the simulation equivalence relation on parity games, which is derived from the simulation preorder as defined in [3]. In this work, this relation is defined as a graph game.

The simulation equivalence relation is weaker than the often used bisimulation relation, so structures that are reduced with this relation as a basis are generally smaller than when using bisimulation. Furthermore, simulation equivalence preserves solutions to parity games, making it very practical for this purpose.

In order to define a strategy for this reduction, we use an approach that is very similar to the theory for simulation equivalence reduction on Kripke structures (simple transition systems with labels in the states) as presented in [1]. We extend this approach to the slightly more complex notion of simulation equivalence on parity games.

To this end, we first minimize the structure with respect to the number of vertices, by generating a quotient structure. Quotient structures are generated from the original structure by merging all equivalent states. It will be shown that the resulting structure can be further minimized by trimming edges to little brothers. Edges to little brothers are defined such that there is always another edge that can replace it, without affecting the simulation equivalence property of the structure.

In Section 2, parity games are introduced, as well as the simulation relation on these structures. Section 3 presents an alternative, but equivalent definition of this relation and some basic properties are proved using this definition. In Section 4, quotient structures are introduced as a means of reducing a parity game. Then, in Section 5, the notion of little brothers is defined for parity games and using this definition, the reduction strategy is extended. Lastly, Section 6 gives an approach for finding a unique minimal parity game with respect to simulation equivalence.

2 Preliminaries

2.1 Parity Games

A parity game is a graph game, played by two players, named *Even* (represented by $\diamond$) and *Odd* (represented by $\Box$). A parity game is played on a game graph, where each vertex is owned by one of the players (represented graphically by the shape of the vertex) and each vertex has some natural number, called *priority*, associated to it.

Definition 2.1 (Parity Game) *A parity game G is a graph game played on a game graph $\mathcal{G} = (V, \rightarrow, p, (V_\diamond, V_\Box))$ where V is the set of vertices of the graph, $\rightarrow \subseteq V \times V$ is a total edge relation, $(V_\diamond, V_\Box)$ is a partitioning of V such that $V_\diamond$ and $V_\Box$ are the sets of vertices owned by player* Even *and* Odd *respectively and $p : V \rightarrow N$ is a priority function, assigning a natural number to each vertex.*

Figure 1 gives an example of a parity game.

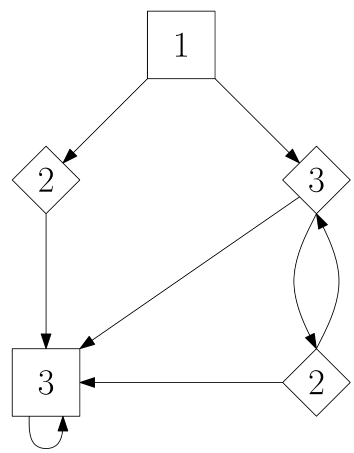

Figure 1: Example of a game graph of a parity game. It is possible for player *Odd* to win from the topmost vertex by moving the token left.

A parity game is played by placing a token on an initial vertex and moving the token through the graph, following the edge relation $\rightarrow$. The token may only be moved by the owner of the vertex on which the token lies.

A play π in a parity game is an infinite path $\langle v_1, v_2, v_3, ...\rangle$ through the game graph. Assume a parity game G and a play π. Define $Inf(\pi)$ as the set of priorities which occur infinitely often in the vertices in π. Play π is won by player *Even* if and only if the lowest value in $Inf(\pi)$ is even. Otherwise, player *Odd* wins.

A strategy for player P is a partial function $\rho_P : V^* \times V_P \rightarrow V$ that decides for vertices owned by P the vertex the token is played to based on the history of the vertices that has been visited.

A play π is consistent with a strategy ρ if and only if for all paths $\pi[0..n]$ that are in the domain of ρ it holds that $\pi[n + 1] = \rho(\pi[0..n])$.

A strategy ρ_P is winning from a vertex v if and only if all consistent plays from v are winning for player P. If and only if a winning strategy exists for player *Even* from the vertex representing the desired property, does the property hold on the model.

$v_1 \in$	$v_2 \in$	First:	On:	Second:	On:
$V_\diamond$	$V_\diamond$	S	v_1	D	v_2
$V_\diamond$	$V_\Box$	S	v_1	S	v_2
$V_\Box$	$V_\diamond$	D	v_2	D	v_1
$V_\Box$	$V_\Box$	S	v_2	D	v_1

Table 1: A table of admissible moves in the simulation game. A table row represents a turn. The first two columns describe the situation which holds at the start of the turn and the two latter columns describe the two subsequent steps that are done in the turn, by stating the player (D or S) and the vertex from which they may move a token.

2.2 Simulation Relation on Parity Games

A notion of simulation relation on parity games is assumed as defined in [3]. This definition is based on a simulation game, which is a turn-based game with two players, named *Duplicator* (D) and *Spoiler* (S). The simulation game of a parity game $G = (V, \rightarrow, p, (V_\diamond, V_\Box))$ is played on a graph $(V \times V, \rightarrow_s)$ where $\rightarrow_s \subseteq (V \times V) \times (V \times V)$ is defined such that $((v_1, v_2), (v_1', v_2')) \in \rightarrow_s$ if and only if $v_1 \rightarrow v_1'$ and $v_2 \rightarrow v_2'$. A play π on this graph is won by *Duplicator* if for all vertices $(v_1, v_2) \in \pi$ it holds that $p(v_1) = p(v_2)$.

A move in the game graph from vertex (v_1, v_2) is done according to Table 1.

Figure 2 demonstrates a play in a simulation game.

A strategy $\rho : V \times V \rightarrow V \times V$ for player P is a partial function which for each combination of parity game vertices in which P can do a step, gives the result of the step done by P. Note that each turn in the simulation game consists of two such steps.

Definition 2.2 (Strategy) *A partial function $\rho : V \times V \rightarrow V \times V$ is a strategy for player P for the simulation game of G if for all pairs of parity game vertices $(v_1, v_2) \in V \times V$ such that P may make a step from (v_1, v_2), there exists a pair $(v_1', v_2') \in V \times V$ such that $((v_1, v_2), (v_1', v_2')) \in \rho$ and $v_1 \rightarrow v_1' \wedge v_2 = v_2'$ if P is allowed to take a step on v_1 and $v_2 \rightarrow v_2' \wedge v_1 = v_1'$ if P is allowed to take a step on v_2.*

Definition 2.3 (Consistent play) *A play π is consistent with strategy ρ_P for player P if for all $v_1, v_2 \in V$ and $n \in \mathbb{N}$ such that $(v_1, v_2) = \pi[n]$, the properties in Table 2 hold for (v_1', v_2') such that $(v_1', v_2') = \pi[n+1]$.*

First:	On:	Second:	On:	Property:
P	v_1	Q	v_2	$\rho_P((v_1, v_2)) = (v_1', v_2)$
P	v_2	Q	v_1	$\rho_P((v_1, v_2)) = (v_1, v_2')$
Q	v_1	P	v_2	$\rho_P((v_1', v_2)) = (v_1', v_2')$
Q	v_2	P	v_1	$\rho_P((v_1, v_2')) = (v_1', v_2')$
P	v_1	P	v_2	$\rho_P((v_1, v_2)) = (v_1', v_2) \wedge \rho_P((v_1', v_2)) = (v_1', v_2')$
P	v_2	P	v_1	$\rho_P((v_1, v_2)) = (v_1, v_2') \wedge \rho_P((v_1, v_2')) = (v_1', v_2')$
Q	v_1	Q	v_2	*true*
Q	v_2	Q	v_1	*true*

Table 2: The properties that have to hold for $\pi[n+1] = (v_1', v_2')$ in a play π such that $\pi[n] = (v_1, v_2)$, in order for π to be consistent with strategy ρ_P for player P. The first two columns describe the turn in the game, and correspond to Table 1. We abstract from the exact players, and call them P and Q.

Definition 2.4 (Winning Strategy) *A strategy ρ for player P is winning if all consistent plays π are winning for P.*

Definition 2.5 (Simulation) *Let $G = (V, \to, p, (V_\diamond, V_\Box))$ be a parity game. A vertex $v_1 \in V$ is simulated by vertex $v_2 \in V$, if player* Duplicator *has a winning strategy for the simulation game starting in (v_1, v_2).*

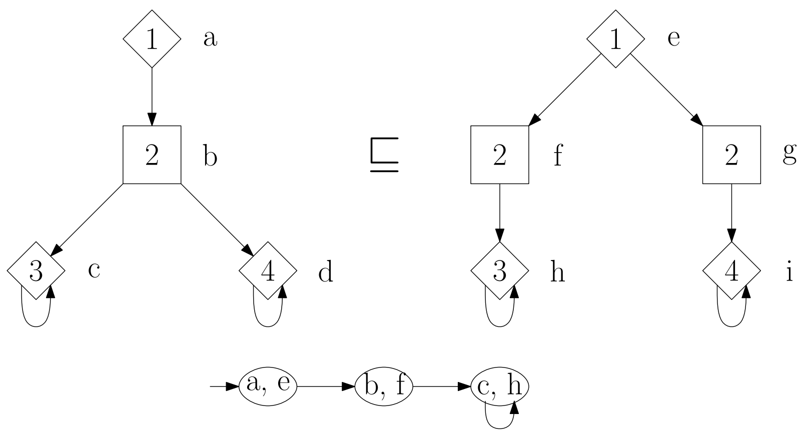

Figure 2: Two parity games, the right one of which simulates the left one. Below, a play in the game graph starting from (a, e) is shown, on which player *Duplicator* wins. It can be seen that vertices in the game graph represent a pair of vertices in the original games. Moves are done according to Table 1.

Definition 2.6 (Simulation Equivalence) *Let $G = (V, \to, p, (V_\diamond, V_\Box))$ be a parity game. A vertex $v_1 \in V$ is simulation equivalent to a vertex $v_2 \in V$, if v_1 simulates v_2 and v_2 simulates v_1.*

3 Simulation

While the simulation game provides an unambiguous definition of simulation, an alternative definition is needed to facilitate reasoning about the simulation relation on parity games. The simulation relation on parity games can be formalized, using the simulation game as a basis.

Definition 3.1 (Simulation Relation) *Let $G = (V, \to, p, (V_\diamond, V_\Box))$ be a parity game. Relation $H \subseteq V \times V$ is a simulation relation if and only if the following properties hold for all v_1, v_2 such that $(v_1, v_2) \in H$:*

- $p(v_1) = p(v_2)$
- *The properties described in Table 3.*

$v_1 \in$	$v_2 \in$	Property:
$V_\diamond$	$V_\diamond$	$(\forall v_1' : v_1 \to v_1' : (\exists v_2' : v_2 \to v_2' : (v_1', v_2') \in H))$
$V_\diamond$	$V_\Box$	$(\forall v_1' : v_1 \to v_1' : (\forall v_2' : v_2 \to v_2' : (v_1', v_2') \in H))$
$V_\Box$	$V_\diamond$	$(\exists v_2' : v_2 \to v_2' : (\exists v_1' : v_1 \to v_1' : (v_1', v_2') \in H))$
$V_\Box$	$V_\Box$	$(\forall v_2' : v_2 \to v_2' : (\exists v_1' : v_1 \to v_1' : (v_1', v_2') \in H))$

Table 3: The properties that have to hold for all pairs of vertices (v_1, v_2) in a simulation relation H. The first two columns represent a case distinction based on the owners of v_1 and v_2 and the third column gives the property which has to hold in the case which is represented by the row.

It is possible to extend this notion of simulation to a notion of simulation between two games $G_1 = (V_1, \to_1, p_1, (V_{1,\diamond}, V_{1,\Box}))$ and $G_2 = (V_2, \to_2, p_2, (V_{2,\diamond}, V_{2,\Box}))$, by using the presented solution on the merged parity game $(V_1 \cup V_2, \to_1 \cup \to_2, p_1 \cup p_2, (V_{1,\diamond} \cup V_{2,\diamond}, V_{1,\Box} \cup V_{2,\Box}))$.

Definition 3.2 **($\lessdot$)** *Vertex v_2 in a parity game simulates vertex v_1 (written as $v_1 \lessdot v_2$) if and only if there exists a simulation relation H such that $(v_1, v_2) \in H$.*

Definition 3.3 **($\sim$)** *Vertex v_1 and v_2 in a parity game are simulation equivalent (written as $v_1 \sim v_2$) if and only if $v_1 \lessdot v_2$ and $v_2 \lessdot v_1$.*

In order to use this alternative definition of the simulation relation, Definition 2.5 must be proved to be equivalent to Definition 3.2. This is done by relating some simulation relation H to a winning strategy for player *Duplicator*.

Lemma 3.4 *Let $G = (V, \rightarrow, p, (V_\diamond, V_\Box))$ be a parity game. A winning strategy from vertex (v_1, v_2) in the simulation game exists if and only if there exists a simulation relation H such that $(v_1, v_2) \in H$.*

Proof. First, it is proved that given a winning strategy ρ for player *Duplicator* from vertex (v_1, v_2), a simulation relation H exists such that $(v_1, v_2) \in H$.

Define a relation H as the smallest relation such that $(v_1, v_2) \in H$ and for each turn $(v_1', v_2') \rightarrow (v_1'', v_2'')$ which is consistent with ρ, such that $(v_1', v_2') \in H$, it holds that $(v_1'', v_2'') \in H$. It will be proved that H is a simulation relation.

Because ρ is a winning strategy for D, the first property of simulation relations is trivially proved, because all plays which are consistent with ρ are winning for D, so it is not possible to reach a vertex (v_1', v_2') such that $p(v_1') \neq p(v_2')$.

Assume a vertex $(v_1', v_2') \in H$. We make a case distinction on the owners of v_1' and v_2'. Only the case $v_1', v_2' \in V_\diamond$ is elaborated, as the other three cases are very similar.

In this case, player *Spoiler* may first do a step on v_1, and *Duplicator* follows with a step on v_2. Because $(v_1', v_2') \in H$, it holds that $(\forall v_1'' : v_1' \rightarrow v_1'' : (\exists v_2'' : v_2' \rightarrow v_2'' : (v_1'', v_2'') \in H))$. By definition of the strategy, for all such v_1'' there is a v_2'' such that $((v_1'', v_2') \rightarrow (v_1'', v_2'')) \in \rho$. Thus, again by definition, the results of these steps (for all v_1'') are included in H.

Secondly, it will be proved that given a simulation relation H with $(v_1, v_2) \in H$, a winning strategy ρ for D starting from (v_1, v_2) exists. To this end, we will prove that for all (v_1', v_2') there is a possibility for *Duplicator* to lead the game to a vertex (v_1'', v_2'') such that $(v_1'', v_2'') \in H$. These steps then form strategy ρ, which is trivially winning, since by definition, for all $(v_1', v_2') \in H$, it holds that $p(v_1) = p(v_2)$.

Only the case in which $v_1' \in V_\diamond$ and $v_2' \in V_\diamond$ is elaborated. The other three cases are very similar.

In this case, player S makes the first step in v_1'. Assume some v_1'' such that S makes the step in the left game to v_1''. Then *Duplicator* makes a step in v_2'. Assume v_2'' such that $v_2' \rightarrow v_2''$ and $(v_1'', v_2'') \in H$. By definition of H, such a v_2'' exists for all possibilities of v_1'' (reading from the first case in the table in Definition 3.2). Thus, such a step is possible. □

Lemma 3.5 *Let $G = (V, \rightarrow, p, (V_\diamond, V_\Box))$ be a parity game. Relation $\lessdot$ is the largest simulation relation.*

Proof. Assume simulation relation I such that $|I| > |\lessdot|$. Then assume $(v_1, v_2) \in (I - \lessdot)$. By definition of $\lessdot$, there is no simulation relation H such that $(v_1, v_2) \in H$. Thus I is no simulation relation, since $(v_1, v_2) \in I$.

The relation $\lessdot$ is clearly a simulation relation. All pairs of related vertices (v_1, v_2) have equal priorities and the properties in the table hold for all related tuples, since (v_1, v_2) are related by some simulation relation H. Take this H. Then all properties in the table hold for this H. □

3.1 Properties

In order to use this simulation relation, a few properties about this relation need to the proved in order to facilitate reasoning about the relation. In this section, it will be shown that this relation is a preorder, by proving reflexivity and transitivity.

Lemma 3.6 (Reflexivity) *Assume parity game vertex $v \in V$. Then it holds that $v \lessdot v$.*

Proof. Assume a relation $H \subseteq V \times V$ such that $(v_1, v_2) \in H \Leftrightarrow v_1 = v_2$.
Because $v = v$, the first property of simulation relations trivially holds.
For the second property, only the first and last case have to be considered for v, because the owner of node v equals the owner of node v. These properties are symmetrical, so only the first property will be explicitly proved. Assume some v_1' such that $v \to v_1'$. Then there must be some v_2' such that $v \to v_2'$ and $(v_1', v_2') \in H$. Taking $v_2' = v_1'$ trivially validates the property. □

Lemma 3.7 (Transitivity) *Assume a parity game $G = (V, \to, p, (V_\diamond, V_\Box))$. For all $(v_1, v_2, v_3) \in V \times V \times V$ such that $v_1 \lessdot v_2$ and $v_2 \lessdot v_3$, it holds that $v_1 \lessdot v_3$.*

Proof. By definition of simulation, there exist simulation relations $H_{1,2} \subseteq V \times V$ and $H_{2,3} \subseteq V \times V$ where $(v_1, v_2) \in H_{1,2}$ and $(v_2, v_3) \in H_{2,3}$.
Define $H_{1,3} \subseteq V \times V$ such that $(v_1, v_3) \in H_{1,3}$ if and only if there exists a v_2 such that $(v_1, v_2) \in H_{1,2}$ and $(v_2, v_3) \in H_{2,3}$.
It remains to be proved that $H_{1,3}$ is a simulation relation. By definition of $H_{1,3}$, there exists a v_2 such that $(v_1, v_2) \in H_{1,2}$, so $p(v_2) = p(v_1)$ and since $(v_2, v_3) \in H_{2,3}$, it holds that $p(v_3) = p(v_2) = p(v_1)$, so the first property is validated.
For proving the second property, we assume $(v_1, v_3) \in H_{1,3}$. Then we define v_2 such that $(v_1, v_2) \in H_{1,2}$ and $(v_2, v_3) \in H_{2,3}$. We distinguish the following cases:

- $(v_1, v_2, v_3) \in V_\diamond \times V_\diamond \times V_\diamond$:

  For all v_1' such that $v_1 \to v_1'$ it holds that there exists a v_2' such that $v_2 \to v_2'$ and $(v_1', v_2') \in H_{1,2}$. For this v_2', it also holds that there exists a v_3' such that $v_3 \to v_3'$ and $(v_2', v_3') \in H_{2,3}$. Thus, for all v_1' such that $v_1 \to v_1'$, there exists a v_3' such that $v_3 \to v_3'$ and $(v_1', v_3') \in H_{1,3}$.

- $(v_1, v_2, v_3) \in V_\diamond \times V_\diamond \times V_\Box$:

  For all v_1' such that $v_1 \to v_1'$ it holds that there exists a v_2' such that $v_2 \to v_2'$ and $(v_1', v_2') \in H_{1,2}$. For this v_2', it also holds that for all v_3' such that $v_3 \to v_3'$ it holds that $(v_2', v_3') \in H_{2,3}$. Thus, for all v_1' and v_3' such that $v_1 \to v_1'$ and $v_3 \to v_3'$, it holds that $(v_1', v_3') \in H_{1,3}$.

- $(v_1, v_2, v_3) \in V_\diamond \times V_\Box \times V_\diamond$:

  For all v_1' and v_2' such that $v_1 \to v_1'$ and $v_2 \to v_2'$ it holds that $(v_1', v_2') \in H_{1,2}$. It also holds that there exists a v_3' such that $v_3 \to v_3'$ and $(v_2', v_3') \in H_{2,3}$. Thus, for all v_1' such that $v_1 \to v_1'$, there exists a v_3' such that $v_3 \to v_3'$ and $(v_1', v_3') \in H_{1,3}$.

- $(v_1, v_2, v_3) \in V_\diamond \times V_\Box \times V_\Box$:

  For all v_1' and v_2' such that $v_1 \to v_1'$ and $v_2 \to v_2'$ it holds that $(v_1', v_2') \in H_{1,2}$. It also holds that for all v_3' such that $v_3 \to v_3'$, $(v_2'', v_3') \in H_{2,3}$ for some v_2'' such that $v_2 \to v_2''$. Note that $(v_1', v_2'') \in H_{1,2}$. Thus, for all v_1' such that $v_1 \to v_1'$, for all v_3' such that $v_3 \to v_3'$ it holds that $(v_1', v_3') \in H_{1,3}$. Because in parity games, each vertex has at least one outgoing edge, there exists a v_3' such that this holds.

- $(v_1, v_2, v_3) \in V_\Box \times V_\diamond \times V_\diamond$:

  There exist v_2' and v_1' such that $v_2 \to v_2'$, $v_1 \to v_1'$ and $(v_1', v_2') \in H_{1,2}$. There also exists a v_3' such that $v_3 \to v_3'$ and $(v_2', v_3') \in H_{1,3}$. Therefore, there exist v_1' and v_3' such that $(v_1', v_3') \in H_{1,3}$.

- $(v_1, v_2, v_3) \in V_\Box \times V_\diamond \times V_\Box$:

  There exist v_2' and v_1' such that $v_2 \to v_2'$, $v_1 \to v_1'$ and $(v_1', v_2') \in H_{1,2}$. Furthermore, for all v_3' such that $v_3 \to v_3'$ it holds that $(v_2', v_3') \in H_{2,3}$. For all v_3' such that $v_3 \to v_3'$ it holds that there exists a v_1' such that $v_1 \to v_1'$ and $(v_1', v_3') \in H_{1,3}$.

- $(v_1, v_2, v_3) \in V_\Box \times V_\Box \times V_\diamond$:

  There exists a v_3' such that $v_3 \to v_3'$ for which there exists a v_2' such that $v_2 \to v_2'$ and $(v_2', v_3') \in H_{2,3}$. For all such v_2' it holds that there exists a v_1' such that $v_1 \to v_1'$ and $(v_1', v_2') \in H_{1,2}$. Thus, there exist v_3' and v_1' such that $v_1 \to v_1'$, $v_3 \to v_3'$ and $(v_1', v_3') \in H_{1,3}$.

- $(v_1, v_2, v_3) \in V_\Box \times V_\Box \times V_\Box$:

  For all v_3' such that $v_3 \to v_3'$ it holds that there exists a v_2' such that $v_2 \to v_2'$ and $(v_2', v_3') \in H_{2,3}$. For all such v_2' it holds that there exists a v_1' such that $v_1 \to v_1'$ and $(v_1', v_2') \in H_{1,2}$. Thus, for all v_3' such that $v_3 \to v_3'$, there exists a v_1' such that $v_1 \to v_1'$ and $(v_1', v_3') \in H_{1,3}$.

□

Theorem 3.8 *Simulation relation $\lessdot$ is a preorder.*

Proof. This is a direct consequence of Lemma 3.6 and Lemma 3.7. □

4 Quotient structure

The approach for reduction of parity games using simulation equivalence as a basis, is defining a quotient structure of a parity game. The defined quotient structure is similar to the universal quotient structure on Kripke structures as given in [1], but extended to the realm of parity games.

Definition 4.1 (Quotient Structure) *Quotient structure $M_q = (\Sigma, \to_q, p_q, (\Sigma_\diamond, \Sigma_\Box))$ of a parity game $G = (V, \to, p, (V_\diamond, V_\Box))$ is defined as follows:*

- *Σ is the set of all equivalence classes with respect to the simulation equivalence of parity games.*
- *$(\Sigma_\diamond, \Sigma_\Box)$ is a partitioning of Σ, such that $\Sigma_\diamond$ is a set of equivalence classes with respect to the simulation equivalence of parity games, which contain at least one vertex v such that $v \in V_\diamond$ and $\Sigma_\Box$ is a set of equivalence classes which contain at least one vertex v such that $v \in V_\Box$.*
- $\to_q = \{(\alpha_1, \alpha_2) \in \Sigma \times \Sigma | (\forall v_1 \in \alpha_1 : (\exists v_2 \in \alpha_2 : v_1 \to v_2))\}$
- $p_q = \{(\alpha, n) | (\forall v \in \alpha : p(v) = n)\}$

  (note that all $v \in \alpha$ have equal priorities)

Note that equivalence classes may contain both even and odd nodes. Such classes can be arbitrarily put into $\Sigma_\diamond$ or $\Sigma_\Box$. In addition to this, we denote the set of all such classes as $\Sigma_\circ$.

Figure 3 shows an example of the quotient structure of a parity game.

4.1 Properties

Before it can be proved that the defined structure is simulation equivalent to the original parity game, some properties of this structure must be proved. It will be shown that mixed even/odd equivalence classes have exactly one outgoing edge (with $\to_q$ as the transition relation).

Furthermore, the quotient structure is proved to be a valid parity game. This is done by showing that the transition relation $\to_q$ is total. Showing that the quotient structure is a valid parity game is necessary, because the used simulation relation is defined only on parity games.

Lemma 4.2 *Assume parity game quotient structure $M_q = (\Sigma, \to_q, p_q, (\Sigma_\diamond, \Sigma_\Box))$. Transition relation $\to_q$ is total.*

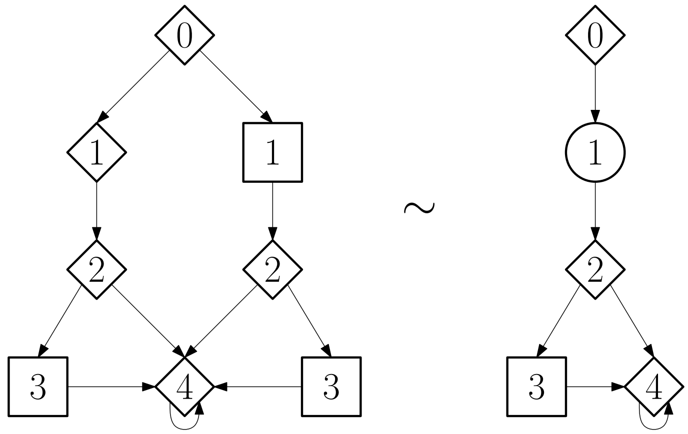

Figure 3: An example of the quotient structure of a parity game. The right game is a quotient structure of the left game. A circle is used to indicate that the equivalence class represented by the vertex contains even and odd vertices. This vertex can be owned by either player. The quotient structure was obtained by calculating the simulation equivalence relation within the left game, and merging the related vertices, merging the edges as defined in Definition 4.1. In this case, all vertices with equal priorities are simulation equivalent. The left game is simulation equivalent to its quotient structure.

Proof. Assume $\rightarrow_q$ is not total. Then there must exist some $\alpha \in \Sigma$ such that there exists no $\alpha' \in \Sigma$ such that $\alpha \rightarrow_q \alpha'$. Take this α. Then $(\forall \alpha' \in \Sigma : (\alpha, \alpha') \notin \rightarrow_q)$. Thus, by definition of $\rightarrow_q$, for all $\alpha' \in \Sigma$ it holds that $(\exists v \in \alpha : (\forall v' \in \alpha' : v \not\rightarrow v'))$. Take this v. Because parity games are total, there exists a $v' \in V$ such that $v \rightarrow v'$. Take such a v'. It holds that $v' \notin \alpha'$ for all $\alpha' \in \Sigma$. This can not hold, because $v' \sim v'$ (by reflexivity of $\lessdot$) and therefore there exists an equivalence class $\alpha' \in \Sigma$ such that $v' \in \alpha'$. By contradiction, $\rightarrow_q$ is total. □

Theorem 4.3 *Quotient structures of parity games are parity games.*

Proof. This follows directly from Lemma 4.2. □

Lemma 4.4 *Assume parity game quotient structure $M_q = (\Sigma, \rightarrow_q, p_q, (\Sigma_\diamond, \Sigma_\Box))$ and $\Sigma_\circ$ the set of mixed even/odd equivalence classes. For every pair of successors α'_1, α'_2 of vertex $\alpha \in \Sigma_\circ$, it holds that $\alpha'_1 \sim \alpha'_2$.*

Proof. By definition of $\rightarrow_q$, it holds that all $v \in \alpha$ must have a successor in α'_1 and one in α'_2. Assume $v_1, v_2 \in \alpha$ such that $v_1 \in V_\diamond$ and $v_2 \in V_\Box$ (which is possible since $\alpha \in \Sigma_\circ$). Since $v_1 \lessdot v_2$, it holds that $(\forall v'_1, v'_2 : v_1 \rightarrow v'_1 \wedge v_2 \rightarrow v'_2 : v'_1 \lessdot v'_2)$. Pick v'_1 and v'_2 such that $v'_1 \in \alpha'_1$ and $v'_2 \in \alpha'_2$. (This is allowed by definition of $\rightarrow_q$.) Because of transitivity of the simulation relation, it holds that $\alpha'_1 \lessdot \alpha'_2$.

Symmetrically, v'_3 and v'_4 can be picked such that $v_1 \rightarrow v'_3$, $v_2 \rightarrow v'_4$, $v'_4 \in \alpha'_1$ and $v'_3 \in \alpha'_2$. It then holds that $v'_3 \lessdot v'_4$ and by extension $\alpha'_2 \lessdot \alpha'_1$.

Thus, $\alpha'_1 \sim \alpha'_2$. □

As a consequence of this, equivalence classes $\alpha \in \Sigma_\circ$ have at most one outgoing edge.

4.2 Correctness

In order for the quotient structure to represent a reduced version of the parity game with respect to the simulation equivalence relation, it has to hold that the resulting structure is a parity game (which follows from Lemma 4.2) and the structure must be simulation equivalent to the original parity game.

Lemma 4.5 *Let $M_q = (\Sigma, \rightarrow_q, p_q, (\Sigma_\diamond, \Sigma_\Box))$ be the quotient structure of parity game $G = (V, \rightarrow, p, (V_\diamond, V_\Box))$ Then it holds that $G \lessdot M_q$.*

Proof. First, recall that $\lessdot \subseteq V \times V$ is the largest simulation relation.

Let $H^{sq} \subseteq V \times \Sigma$ be the relation $\{(v_1, \alpha) | (\exists v_2 \in \alpha : v_1 \lessdot v_2)\}$. We prove that H^{sq} is a simulation relation. Let $\Sigma_\circ$ be the set of mixed even/odd equivalence classes. The definition of simulation relations used is the one on parity games and all $\alpha \in \Sigma_\circ$ are assumed to have an arbitrary owner ($\diamond$ or $\Box$).

First it is proved that $(\forall (v_1, \alpha) \in H^{sq} : p(v_1) = p_q(\alpha))$. Assume $(v_1, \alpha) \in H^{sq}$. Then there exists a $v_2 \in \alpha$ such that $v_1 \lessdot v_2$. Because $\lessdot$ is a simulation relation, $p(v_1) = p(v_2)$ and since $v_2 \in \alpha$, $p(v_1) = p_q(\alpha)$.

The second property that needs to be proved depends on the owners of the nodes v_1 and α such that $(v_1, \alpha) \in H^{sq}$. There are four cases:

- $(v_1, \alpha) \in V_\diamond \times (\Sigma_\diamond - \Sigma_\circ)$:

  We show that $(\forall v_1' : v_1 \rightarrow v_1' : (\exists \alpha' : \alpha \rightarrow_q \alpha' : (v_1', \alpha') \in H^{sq}))$. Assume a v_1' such that $v_1 \rightarrow v_1'$. We distinguish two cases:

  - $v_1 \in \alpha$:

    It remains to be proved that there exists an α' such that $\alpha \rightarrow_q \alpha'$ and $(\exists v_2' \in \alpha' : v_1' \lessdot v_2')$. Let I be the set of vertices $\{v_2' \in V | (\exists v_2 \in \alpha : v_2 \rightarrow v_2' \wedge v_1' \lessdot v_2')\}$. Note that I is nonempty, because $v_1' \in I$. Let $v_m' \in I$ such that all vertices $v_n' \in I$ that simulate v_m' are simulation equivalent to v_m'. In other words, v_m' is a maximal vertex in I with respect to simulation. Then assume v_m such that $v_m \in \alpha$ and $v_m \rightarrow v_m'$. Define α_1' such that $v_m' \in \alpha_1'$. It will be proved that $\alpha \rightarrow_q \alpha_1'$, proving that there exists an α' which adheres to the requirements. To this end, it will be shown that for every vertex $v_3 \in \alpha$, there exists a successor $v_3' \in \alpha_1'$. By definition of $\rightarrow_q$, this implies that $\alpha \rightarrow_q \alpha_1$.

    Because $v \in \alpha$ and $v_m \in \alpha$, it holds that $v_m \lessdot v_3$ and $v_3 \in V_\diamond$ and $v_m \in V_\diamond$. This implies that there exists a v_3' such that $v_3 \rightarrow v_3'$ and $v_m' \lessdot v_3'$. Because $v_m' \in I$, it holds that $v_1' \lessdot v_m'$ and by transitivity of simulation, $v_1' \lessdot v_3'$.

  - $v_1 \notin \alpha$:

    Assume some $v_2 \in \alpha$ such that $v_1 \lessdot v_2$. Then, since $v_2 \in V_\diamond$, there must exist a v_2' such that $v_1' \lessdot v_2'$. The previous case ($v_1 \in \alpha$) implies that there exists an α' such that $\alpha \rightarrow_q \alpha'$ and $(v_2', \alpha') \in H^{sq}$. Assume such an α'. Since $(v_2', \alpha') \in H^{sq}$, there exists a $v_3' \in \alpha'$ such that $v_2' \lessdot v_3'$. Since $v_1' \lessdot v_2'$ and $v_2' \lessdot v_3'$, by transitivity of simulation it holds that $v_1' \lessdot v_3'$. Thus, since $v_3' \in \alpha'$, it holds that $(v_1', \alpha') \in H^{sq}$.

  Note that the proof for this case is analogous to the proof given in [1].

- $(v_1, \alpha) \in V_\diamond \times \Sigma_\circ$:

  Depending on the (arbitrarily chosen) owner of node v_1, one of the following two properties must hold:

  - $(\forall v_1' : v_1 \rightarrow v_1' : (\exists \alpha' : \alpha \rightarrow_q \alpha' : (v_1', \alpha') \in H^{sq}))$
  - $(\forall v_1' : v_1 \rightarrow v_1' : (\forall \alpha' : \alpha \rightarrow_q \alpha' : (v_1', \alpha') \in H^{sq}))$

  Both properties are proved independently of the owner of α.

  Because $\alpha \in \Sigma_\circ$, there is exactly one α' such that $\alpha \rightarrow_q \alpha'$. Thus, both statements are equivalent and only the first one will be proved.

  It remains to be proved that for all v_1' such that $v_1 \rightarrow v_1'$ there exists an α' such that $\alpha \rightarrow_q \alpha'$ and there exists a $v_2' \in \alpha'$ such that $v_1' \lessdot v_2'$. By picking α' such that $v_1' \in \alpha'$ and picking v_2' such that $v_1' = v_2'$, it remains to be proved that $\alpha \rightarrow_q \alpha'$.

  This is true if and only if $(\exists v_1' \in \alpha' : v_1 \rightarrow v_1')$, which is trivially true, because the edge relation of a parity game is total.

- $(v_1, \alpha) \in V_\Box \times (\Sigma_\Box - \Sigma_\circ)$:

  We show that $(\forall \alpha' : \alpha \to_q \alpha' : (\exists v_1' : v_1 \to v_1' : (v_1', \alpha') \in H^{sq}))$.

  Assume an α' such that $\alpha \to_q \alpha'$. Then there must exist some v_1' and v_2' such that $v_1 \to v_1'$, $v_2' \in \alpha'$ and $v_1' \lessdot v_2'$.

  By definition of $\to_q$, it holds that $(\forall v_1 \in \alpha : (\exists v_1' \in \alpha' : v_1 \to v_1'))$. Instantiating with v_1 yields existence of a v_1' such that $v_1 \to v_1'$ and $v_1' \in \alpha'$. Now assume a v_2' such that $v_2' = v_1'$. Then it holds that $v_2' \in \alpha'$ and $v_1' \lessdot v_2'$ (by reflexivity of $\lessdot$).

- $(v_1, \alpha) \in V_\Box \times \Sigma_\circ$:

  As in the second case, there are two properties of which one must hold, depending on the owner of α:

  - $(\exists \alpha' : \alpha \to_q \alpha' : (\exists v_1' : v_1 \to v_1' : (v_1', \alpha') \in H^{sq}))$
  - $(\forall \alpha' : \alpha \to_q \alpha' : (\exists v_1' : v_1 \to v_1' : (v_1', \alpha') \in H^{sq}))$

  The two properties will be proved, independently on the owner of α. Because $\alpha \in \Sigma_\circ$, it has exactly one outgoing edge, so the two properties are equivalent. The proof for the second property is given in the second case, so it is omitted here.

The cases $(v_1, \alpha) \in V_\diamond \times (\Sigma_\Box - \Sigma_\circ)$ and $(v_1, \alpha) \in V_\Box \times (\Sigma_\diamond - \Sigma_\circ)$ are not evaluated, because they can not exist by definition of the relation H^{sq}. □

Lemma 4.6 *Let $M_q = (\Sigma, \to_q, p_q, (\Sigma_\diamond, \Sigma_\Box))$ be the quotient structure of parity game $G = (V, \to, p, (V_\diamond, V_\Box))$. Then it holds that $M_q \lessdot G$.*

Proof. Define $H^{qs} \subseteq M_q \times G$ as $\{(\alpha, v_1) | v_1 \in \alpha\}$. We prove that H^{qs} is a simulation relation.

First it is proved that $(\forall (\alpha, v_1) \in H^{qs} : p(v_1) = p_q(\alpha))$. Assume $(\alpha, v_1) \in H^{qs}$. Then $v_1 \in \alpha$. By definition of $p_q(\alpha)$ it holds that $p_q(\alpha) = p(v_1)$.

The second property that needs to be proved depends on the owners of the nodes $(\alpha, v_1) \in H^{qs}$. Let $\Sigma_\circ$ be the set of mixed even/odd equivalence classes.

There are four cases:

- $(\alpha, v_1) \in (\Sigma_\diamond - \Sigma_\circ) \times V_\diamond$:

  We show that $(\forall \alpha' : \alpha \to_q \alpha' : (\exists v_1' : v_1 \to v_1' : (\alpha', v_1') \in H^{qs}))$. Assume some α' such that $\alpha \to_q \alpha'$. Then by definition of $\to_q$, it holds that $(\exists v_1' \in \alpha' : v_1 \to v_1')$. Assume such a v_1'. By definition of H^{qs}, $(\alpha', v_1') \in H^{qs}$.

  Note that this case is analogous to the one given in [1].

- $(\alpha, v_1) \in \Sigma_\circ \times V_\diamond$:

  Depending on the owner of α, one of the following two properties must hold.

  - $(\forall \alpha' : \alpha \to_q \alpha' : (\exists v_1' : v_1 \to v_1' : (\alpha', v_1') \in H^{qs}))$
  - $(\exists v_1' : v_1 \to v_1' : (\exists \alpha' : \alpha \to_q \alpha' : (\alpha', v_1') \in H^{qs}))$

  Both properties are equivalent because α has exactly one outgoing state. Therefore, it will only be proved that $(\forall \alpha' : \alpha \to_q \alpha' : (\exists v_1' : v_1 \to v_1' : (\alpha', v_1') \in H^{qs}))$.

  Let α' be a vertex such that $\alpha \to_q \alpha'$. It must be proved that there exists a v_1' such that $v_1 \to v_1'$ and $v_1' \in \alpha'$. By definition of $\to_q$, this is trivially true.

- $(\alpha, v_1) \in (\Sigma_\Box - \Sigma_\circ) \times V_\Box$:

  We show that $(\forall v_1' : v_1 \to v_1' : (\exists \alpha' : \alpha \to_q \alpha' : (\alpha', v_1') \in H^{qs}))$. Assume some v_1' such that $v_1 \to v_1'$. It must be proved that there exists an α' such that $\alpha \to_q \alpha'$ and $v_1' \in \alpha'$. Define α' such that $v_1' \in \alpha'$. It remains to show that $\alpha \to_q \alpha'$. This is true if and only if $(\exists v_2' \in \alpha' : v_1 \to v_2')$. Vertex v_1' satisfies this.

- $(\alpha, v_1) \in \Sigma_\circ \times V_\Box$:

  Like in the second case, one of the following two properties must hold, depending on the owner of α:

  - $(\forall \alpha' : \alpha \rightarrow_q \alpha' : (\forall v_1' : v_1 \rightarrow v_1' : (\alpha', v_1') \in H^{qs}))$
  - $(\forall v_1' : v_1 \rightarrow v_1' : (\exists \alpha' : \alpha \rightarrow_q \alpha' : (\alpha', v_1') \in H^{qs}))$

  Again, the two requirements are equivalent, because α' has exactly one outgoing state. Therefore, only a proof of the second property suffices. This proof is given in the second case, and will be omitted here.

Again, the cases $(\alpha, v_1) \in (\Sigma_\diamond - \Sigma_\circ) \times V_\Box$ and $(\alpha, v_1) \in (\Sigma_\Box - \Sigma_\circ) \times V_\diamond$ are not evaluated, because they can not exist. □

Theorem 4.7 *Let M_q be the quotient structure of parity game G. Then it holds that $M_q \sim G$.*

Proof. This holds as a consequence of Lemma 4.5 and Lemma 4.6. □

4.3 Minimality

It is shown that a quotient structure $M_q = (\Sigma, \rightarrow_q, p_q, (\Sigma_\diamond, \Sigma_\Box))$ of parity game $G = (V, \rightarrow, p, (V_\diamond, V_\Box))$ is a simulation equivalent parity game to G. Due to the nature of the quotient structure, M_q contains at most the same number of edges and vertices as G, so $|\Sigma| \leqslant |V|$.

This can be easily seen by looking at the containment relation $(\in \subseteq \Sigma \times V)$. This relation is total, because all quotient structure vertices contain at least one vertex from the original parity game. And all vertices from the original parity game are included in exactly one vertex from the quotient structure. A similar argument can be given for the edges. The number of outgoing edges from a quotient structure vertex is at most equal to the sum of the number of outgoing edges from the included vertices.

The resulting quotient structure M_q is not necessarily minimal.

When evaluating a parity game G in which all nodes are owned by player even ($\diamond$), both the definition of simulation equivalence and the definition of the quotient structure coincide with the definitions on Kripke structures (using the $\forall$-quotient), as given in [1]. Here, it was shown that this quotient is not necessarily minimal. This leads to the assumption that the defined quotient structure on parity games is not necessarily minimal. A counterexample demonstrating this, derived by translating a Kripke structure into a parity game in which all vertices are owned by player *Even*, is given in Figure 4. Here, minimality is disproved by removing an additional edge, and reaching a smaller, simulation equivalent quotient structure.

Note that the structure is minimal with respect to the number of nodes. It is not possible to remove any of the nodes without invalidating the simulation equivalence. The reason for this is that the quotient structure contains no simulation equivalent vertices. If it would contain simulation equivalent vertices, those vertices would have been merged into one vertex.

It is not possible to remove any other vertices since for all vertices in the original game, there must be a simulation equivalent counterpart in the reduced game.

5 Little brothers

The previous section showed that quotient structures are not necessarily minimal with respect to the number of edges in the game. In order to further reduce the parity game, unnecessary edges need to be removed. In order to find those edges, we introduce a notion of little brothers in parity games. This notion is similar to the notion of little brothers in Kripke structures, as given in [1].

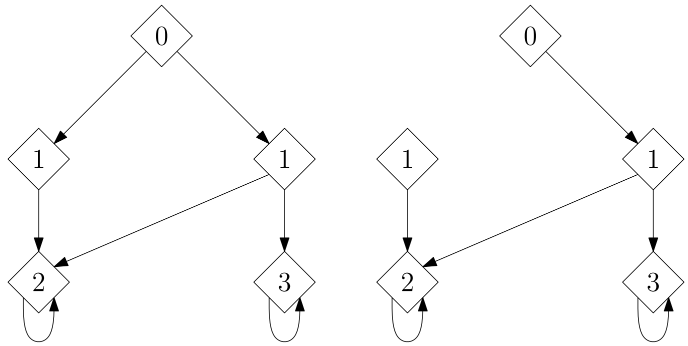

Figure 4: A counterexample for minimality of the quotient structure. The quotient structure is shown on the left and the minimal simulation equivalent parity game is shown on the right.

A little brother is a vertex v' such that there exists a vertex v'' which shares a predecessor v with v', such that if $v \in V_\Box$, v'' simulates v' and if $v \in V_\Diamond$, v' simulates v'' but not $v' \sim v''$. An example of a little brother is shown in Figure 5. It will be shown that it is possible to remove the edge from shared predecessor v to little brother v' without affecting the simulation equivalence relation.

Definition 5.1 *Assume a parity game $G = (V, \to, p, (V_\Diamond, V_\Box))$ and vertices $v, v', v'' \in V$ such that $v \to v'$ and $v \to v''$. Vertex v' is called a little brother of v'' if the properties in Table 4 hold.*

$v \in$	Property:
$V_\Diamond$	$v' \lessdot v'' \wedge \neg(v'' \lessdot v')$
$V_\Box$	$\neg(v' \lessdot v'') \wedge v'' \lessdot v'$

Table 4: The properties which have to hold for v' to be a little brother of v'' with shared predecessor v. The first column represents a case distinction based on the owner of v.

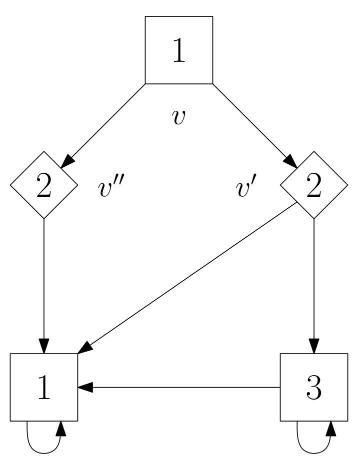

Figure 5: A quotient structure containing a little brother. Vertex v' is a little brother of v'' and v is their shared predecessor.

Note that removing an edge to a little brother will never cause a situation in which a vertex has no outgoing edges, because only one outgoing edge from the shared predecessor will be removed if at least two are present.

Lemma 5.2 *Let $G = (V, \to, p, (V_\diamond, V_\Box))$ be a parity game. Take vertices $v, v', v'' \in V$ such that $v \to v'$, $v \to v''$ and v' is a little brother of v''. Define $G' = (V', \to', p', (V'_\diamond, V'_\Box))$ such that $V'_\diamond = V_\diamond$, $V'_\Box = V_\Box$, $\to' = (\to - \{(v, v')\})$, $p' = p$. Then $G \sim G'$.*

Proof. First, it is proved that $G \lessdot G'$.

Assume $H = \{(v_1, v_3) | v_1 \in V \wedge v_3 \in V' \wedge (\exists v_2 \in V : v_1 = v_2 \wedge v_2 \lessdot v_3)\}$. It is to be proved that H is a simulation relation.

The first property, namely that for all $(v_1, v_3) \in H$ it must hold that $p(v_1) = p'(v_3)$, is trivially proved by stating that by definition of H, $v_1 \lessdot v_3$.

For the second property, take some $v_1 \in V, v_2, v_3 \in V'$ such that $v_1 = v_2$ and $v_2 \lessdot v_3$.

We make a case distinction:

- Case $v_1 \in V_\diamond$:

  We prove that $(\forall v'_1 : v_1 \to v'_1 : (\exists v'_3 : v_3 \to v'_3 : (v'_1, v'_3) \in H))$. Assume a $v'_1 \in V$ such that $v_1 \to v'_1$.

  We distinguish two cases:

  - Case $v_3 = v \wedge v'_1 = v'$:
    Assume $v'_3 \in V'$ such that $v'_3 = v''$. Then it holds that $v_3 \to v'_3$ and $v'_1 \lessdot v'_3$, so $(v'_1, v'_3) \in H$.
  - Case $v_3 \neq v \vee v'_1 \neq v'$:
    Assume $v'_3 \in V'$ such that $v'_1 = v'_3$. Then it holds that $v'_1 \lessdot v'_3$, so $(v'_1, v'_3) \in H$. Since $v_3 \neq v$ or $v'_3 \neq v'$, it holds that $v_3 \to v'_3$, because the edge was not removed.

- Case $v_1 \in V_\Box$:

  We prove that $(\forall v'_3 : v_3 \to v'_3 : (\exists v'_1 : v_1 \to v'_1 : (v'_1, v'_3) \in H))$. Assume a $v'_3 \in V$ such that $v_3 \to v'_3$.

  Then there exists a v'_1 such that $v'_1 = v'_3$ and thus $v'_1 \lessdot v'_3$. Because $\to' \subset \to$, it also holds that $v_1 \to v'_1$.

Secondly, it is proved that $G' \lessdot G$: Assume $H = \{(v_3, v_1) | v_1 \in V \wedge v_3 \in V' \wedge (\exists v_2 \in V : v_2 = v_3 \wedge v_2 \lessdot v_1)\}$. It is to be proved that H is a simulation relation.

Again, the first property trivially holds, since for all $(v_3, v_1) \in H$ it holds that $v_3 \lessdot v_1$.

For the second property, take some $v_1, v_2 \in V$ and $v_3 \in V'$ such that $v_2 = v_3$ and $v_2 \lessdot v_1$.

We make another case distinction:

- Case $v_1 \in V_\diamond$:

  We prove that $(\forall v'_3 : v_3 \to v'_3 : (\exists v'_1 : v_1 \to v'_1 : (v'_3, v'_1) \in H))$. Assume a $v'_3 \in V$ such that $v_3 \to v'_3$.

  Then there exists a v'_1 such that $v'_1 = v'_3$ and thus $v'_3 \lessdot v'_1$. Because $\to' \subset \to$, it also holds that $v_1 \to v'_1$.

- Case $v_1 \in V_\Box$:

  We prove that $(\forall v'_1 : v_1 \to v'_1 : (\exists v'_3 : v_3 \to v'_3 : (v'_3, v'_1) \in H))$. Assume a $v'_1 \in V$ such that $v_1 \to v'_1$.

  There are two cases:

  - Case $v_3 = v \wedge v'_1 = v'$:
    Assume $v'_3 \in V'$ such that $v'_3 = v''$. Then it holds that $v_3 \to v'_3$ and $v'_3 \lessdot v'_1$, so $(v'_3, v'_1) \in H$.
  - Case $v_3 \neq v \vee v'_1 \neq v'$:
    Assume $v'_3 \in V'$ such that $v'_1 = v'_3$. Then it holds that $v'_3 \lessdot v'_1$, so $(v'_3, v'_1) \in H$. Since $v_3 \neq v$ or $v'_3 \neq v'$, it holds that $v_3 \to v'_3$.

□

5.1 Minimality with respect to edges

It is shown that removing edges to little brothers further reduces the quotient structure, without affecting simulation equivalence. In order to show that the resulting structure from this operation is truly minimal, it has to be shown that no further edges can be removed.

Definition 5.3 (Reduced) *Assume a parity game G. Then G is called* reduced *if G contains no simulation equivalent vertices and no little brothers.*

Lemma 5.4 *Assume a reduced parity game $G = (V, \to, p, (V_\diamond, V_\Box))$. Then there exists no edge $r \in \to$ such that $G \sim (V', \to_r, p, (V_\diamond, V_\Box))$ where $V' = V$ and $\to_r = (\to - \{r\})$.*

Proof. Assume an edge $r \in \to$, exists such that $G \sim (V', \to_r, p, (V_\diamond, V_\Box))$ and $\to_r = (\to - \{r\})$. Take this r. Define v_1, v_1' such that $r = (v_1, v_1')$.

Then there exist simulation relations $H \in V \times V'$ and $H' \in V' \times V$ and vertex $v_2 \in V'$ such that $(v_1, v_2) \in H$ and $(v_2, v_1) \in H'$.

There are two cases.

- Case $v_1 = v_2$:

  We distinguish cases based on the owner of v_1:

  - $v_1 \in V_\diamond$:

    It holds that $(\forall v_1' : v_1 \to v_1' : (\exists v_2' : v_2 \to v_2' : (v_1', v_2') \in H))$. By instantiating with v_1', there exists a v_2'' such that $v_2 \to_r v_2''$ and $(v_1', v_2'') \in H$. Define $v_2' \in V'$ such that $v_2' = v_1'$. Note that $v_2' \neq v_2''$ because $v_2 \not\to_r v_2'$. There must be some v_1'' such that $v_1 \to v_1''$ and $(v_2'', v_1'') \in H'$.

    By transitivity of simulation, it holds that $v_1' \lessdot v_1''$. Because G is reduced, it must hold that $v_1' \sim v_1''$ (because otherwise v_1' is a little brother of v_1'', because they have v_1 as a shared predecessor) and thus by transitivity, $v_2'' \lessdot v_1'$, so $v_1' \sim v_2''$. Since $v_2' = v_1'$, it holds that $v_2' \sim v_2''$. Therefore $v_2' = v_2''$, because the game is reduced. This contradicts the statement that $v_2' \neq v_2''$.

  - $v_1 \in V_\Box$:

    This case is symmetric to the case in which $v_1 \in V_\Box$. The reverse relations are used (H is swapped with H') to compensate for the difference in owners of the states.

- Case $v_1 \neq v_2$:

  Assume $v_3 \in V'$ such that $v_3 = v_2$. It holds that $(v_3, v_2) \notin H$ or $(v_2, v_3) \notin H'$, because otherwise $v_1 \sim v_3$ and because G is reduced, it would hold that $v_1 = v_3$, which contradicts the case assumption.

  Since $G \sim (V', \to_r, p, (V_\diamond, V_\Box))$, there must be some $v_4 \in V'$ such that $v_3 \sim v_4$. Then v_4 is simulation equivalent to v_2 without the removed edge r, and because G is reduced, they must be the same vertex, thus $v_2 = v_4$. But since v_2 and v_3 can not be related by either H or H', this contradicts simulation equivalence.

All cases lead to a contradiction. Thus, r can not exist and G is minimal with respect to the number of edges. □

6 Unique structure

A procedure was shown through which we can create a reduced parity game, and this type of parity game was proved to be minimal with respect to both vertices and edges. It is, however, not a unique minimal parity game, when we reason about uniqueness up to isomorphism. The reason for this is that there may be two simulation equivalent parity games with an equal number of vertices and edges, but different players for some vertices. An example is shown in Figure 6.

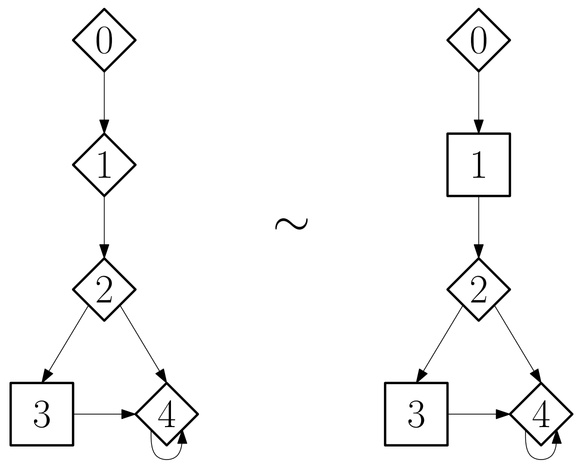

Figure 6: Two different simulation equivalent reduced parity games, proving that reduced parity games are not necessarily unique up to simulation equivalence.

In order to define a unique minimal simulation equivalent structure on parity games, a new notion of minimality must be assumed.

Definition 6.1 **($\sqsubseteq$)** *Assume a total ordering $\sqsubseteq$ on parity games.*
Assume parity games $G = (V, \rightarrow, p, (V_\diamond, V_\Box))$ and $G' = (V', \rightarrow', p', (V'_\diamond, V'_\Box))$. The ordering is defined such that $G \sqsubseteq G'$ if and only if one of the following holds:

- $|V| < |V'|$
- $|V| = |V'| \wedge |\rightarrow| < |\rightarrow'|$
- $|V| = |V'| \wedge |\rightarrow| = |\rightarrow'| \wedge |V_\diamond| \leqslant |V'_\diamond|$

Note that the choice for the ordering by the owner of vertices in Definition 6.1, based on a smaller $|V_\diamond|$, implies a larger $|V_\Box|$, because all vertices are owned by one of both players. The choice for $|V_\diamond|$ over $|V_\Box|$ is arbitrary, and serves only to provide an instance of an ordering based on owners.

6.1 Minimization with respect to owners

Assume a reduced parity game G. Then the game can be minimized to a unique minimum, adhering to the definition of $\sqsubseteq$, by transforming the owners of vertices.

By transforming vertices $v \in V_\diamond$, by removing them from $V_\diamond$ and putting them into $V_\Box$ where allowed, a unique minimal structure is reached.

Transforming vertices in $V_\diamond$ to $V_\Box$ is not always possible.

Theorem 6.2 *Assume a reduced parity game $G = (V, \rightarrow, p, (V_\diamond, V_\Box))$. A vertex $v_1 \in V_\diamond$ is simulation equivalent to a vertex v_2 with the same priority and successors, but owned by player $\Box$ if and only if v_1 has no more than one successor.*

Proof. Assume a vertex $v_1 \in V_\diamond$ and define $succ(v_1)$ as the set of its successors. Now assume a vertex v_2, owned by player $\Box$, with priority $p(v_1)$ and $succ(v_1) = succ(v_2)$.

First, it will be proved that $v_1 \sim v_2$ if $|succ(v_1)| = 1$. The two vertices are simulation equivalent if the following two properties hold:

- $(\forall v'_1 \in succ(v_1) : (\forall v'_2 \in succ(v_2) : v'_1 \lessdot v'_2))$

- $(\exists v'_1 \in succ(v_1) : (\exists v'_2 \in succ(v_2) : v'_2 \lessdot v'_1))$

Since $|succ(v_1)| = |succ(v_2)| = 1$, these properties coincide, and reduce to $v'_1 \sim v'_2$, with $succ(v_1) = \{v'_1\}$ and $succ(v_2) = \{v'_2\}$, which is true, because $succ(v_1) = succ(v_2)$.

It follows from Lemma 4.4 that $|succ(v_1)| = 1$ if $v_1 \sim v_2$, because v_1 and v_2 are part of an equivalence class α with mixed even/odd priorities. Thus, there is only one equivalence class α' such that $\alpha \rightarrow_q \alpha'$. Because G is reduced, there exists only one $v' \in \alpha'$, so $|succ(v'_1)| = 1$. □

Theorem 6.2 shows that nodes owned by player ◇ can be freely changed so they are owned by □, without affecting simulation equivalence, if and only if they have one outgoing edge. Therefore, we can reach a unique minimal simulation equivalent parity game by calculating the quotient structure, reducing it by eliminating edges to little brothers and changing the owner of all nodes with only one outgoing edge to □.

Note that the parity game which is generated by this procedure is necessarily minimal according to Definition 6.1. As shown in Section 4.3, there are no simulation equivalent parity games with fewer vertices, so $|V|$ is minimal. With these vertices, no more edges can be removed without affecting simulation equivalence as shown in Section 5.1, so $|\rightarrow|$ is minimal. Lastly, this section demonstrated a way of minimizing the number of nodes owned by player *Even*. Therefore, the entire game is minimal with respect to $\sqsubseteq$.

7 Conclusion

We have seen a strategy for reduction of a parity game to a minimal parity game which is simulation equivalent to the original game. This reduction strategy, based on a similar reduction on Kripke structures, starts with calculating the quotient structure of the game, thereby minimizing the game with respect to the number of vertices.

As a second step, the game is further reduced by eliminating edges from shared predecessors to little brothers, minimizing the structure with respect to edges.

In order to define a *unique* minimal simulation equivalent parity game, the strategy is extended with the step of changing the owners of all vertices with no more than one outgoing edge to □.

For further research, it would be interesting to extend the existing algorithms for this minimization on Kripke structures, to a minimization on parity games. This could be followed by an evaluation of any possible performance improvements when solving parity games from real-world cases, after first reducing them, thereby determining whether there is any practical gain to be had from this reduction.

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