Correspondence between Kripke Structures and Labeled Transition Systems for Model Minimization

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Abstract

This document is mainly an extension of the work of Michel Reniers and Tim Willemse, which showed the correspondence between Kripke Structures and Labeled Transition Systems for a number of equivalences. They have also established that the embeddings defined by De Nicola and Vaandrager allow us to use minimization techniques modulo bisimulation in one domain, to attain an actual minimization modulo bisimulation in the other domain. In this document, the same property is investigated for simulation equivalence. Extensive proof is given to show that similarity minimization is covered by the embedding.

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1 Introduction

In process theory, many modelling techniques have been developed to describe the behaviour of systems. Two of the most used types of models are *Kripke Structures* and *Labeled Transition Systems*, also denoted by KS and LTS. The difference between these languages is that KS are state-based (meaning that states are labeled with a name), while LTS are event-based (meaning that transitions are labeled).

As the work of De Nicola and Vaandrager has showed, KS and LTS are interchangeable in many contexts. However, it is not very interesting to translate a model from one domain to the other and back. The embeddings between domains are far more useful if we can do actual computations in between, since this means that we can reuse implementations of possibly complex operations. One of the most common operations in proces theory is model minimization. For many reasons, we want our models to be as small as possible, while maintaining the behaviour of the system. In the work by Reniers and Willemse, it is showed that bisimilarity or stuttering equivalence minimization in one domain can be obtained by using an implementation of minimization in the other domain. This document extends their results, by proving that the same property holds for similarity.

Outline. Section 2 gives an overview of the definitions used by the proofs and examples. In section 3, an extensive proof is given that similarity minimization techniques can be shared between KS and LTS, while some examples for this are provided in section 4. Sections 6 and 7 finish the document with conclusions and possible future work.

2 Definitions

For both Kripke Structures and Labeled Transition Systems, this section gives an overview of definitions that are used.

2.1 Kripke Structures

2.1.1 General Definitions

The first step is to define what exactly a Kripke Structure is.

Definition 1 (Kripke Structure)**.** A Kripke Structure K is a 5-tuple $\langle S, I, AP, \rightarrow, L\rangle$, where:

- S is a set of states.
- I is the set of initial states: $I \subseteq S$.
- AP is a set of atomic propositoins.
- $\rightarrow$ is a total transition relation: $\rightarrow \subseteq S \times S$.
- L is a state labelling: $L : S \rightarrow 2^{AP}$.

Note that from now on, the shorthand notation $s \rightarrow t$ is used for stating that $(s, t) \in \rightarrow$.

As the focus of this document is on minimization of models, we need to define the size of a structure in order to compare structures and use minimality.

Definition 2 (Size)**.** The *size* $|K|$ of a Kripke Structure $K = \langle S, I, AP, \rightarrow, L\rangle$ is defined as the number of states it has: $|S|$. This means that, given another Kripke Structure $K' = \langle S', I', AP, \rightarrow', L'\rangle$, we say that $|K| \leq |K'|$ (K is *smaller* than K'), if $|S| \leq |S'|$.

2.1.2 Similarity

When minimizing a KS, we want the resulting KS to be *equivalent* to the original one. Many equivalences have been defined in process theory. In this document, the emphasis is on the equivalence relation called similarity.

Definition 3 (Simulation relation)**.** Let $K = \langle S, I, AP, \rightarrow, L\rangle$ be a Kripke Structure. A relation $B \subseteq S \times S$ is a simulation relation if and only if for every $s, s' \in S$ such that $(s, s') \in B$:

- $L(s) = L(s')$
- For all $t \in S$, if $s \rightarrow t$, then $s' \rightarrow t'$ for some $t' \in S$ such that $(t, t') \in B$.

Definition 4 (Similarity)**.** States $s, s' \in S$ are *similar* (notation: $s \simeq s'$) if and only if there are simulation relations B and B' such that $(s, s') \in B$ and $(s', s) \in B'$.

Definition 5 (Similarity of KS)**.** Two Kripke Structures K and K' are simulation equivalent (or similar), notation $K \simeq K'$, if and only if:

- For all s in I, there is a s' in I', such that $s \simeq s'$.
- For all s' in I', there is a s in I, such that $s' \simeq s$.

2.1.3 Minimization modulo similarity

Using the definitions of the size of a KS and similarity between two KS, we can describe how to perform a minimization. The following definition shows for an arbitrary Kripke Structure, how to compute an equivalent structure that is as small as possible.

Using the notion of simulation equivalence between states, we can define the *equivalence classes* C of a Kripke Structure K. For every maximal set of similar states $s_1, s_2, \ldots, s_n \in S$ (so, $s_1 \simeq s_2 \simeq \cdots \simeq s_n$), there is one corresponding class $c \in C$. We use "[ ]"to denote the class of a state, so if $s \in c$, then $[s] = c$.

Definition 6 ($\forall$-quotient)**.** We can minimize a Kripke Structure K modulo similarity by calculating its $\forall$-quotient structure $K_{/\simeq} = \langle S_{/\simeq}, I_{/\simeq}, AP_{/\simeq}, \rightarrow_{/\simeq}, L_{/\simeq}\rangle$ as defined in [BG03]:

- $S_{/\simeq} = C$, the set of equivalence classes of K
- $I_{/\simeq} = \{c \in C \mid \exists s \in c\ .\ s \in I\}$

- $AP_{/\simeq} = AP$
- $\rightarrow_{/\simeq} = \{(c1, c2) \in C \times C \mid \forall s1 \in c1 \; \exists s2 \in c2 \; . \; s1 \rightarrow s2\}$
- $L_{/\simeq}([s]) = L(s)$

From [BG03], it follows that the following statements hold for $K_{/\simeq}$:

- $K \simeq K_{/\simeq}$
- For every Kripke Structure K' such that $K \simeq K'$, we have that $|K_{/\simeq}| \leq |K'|$.

2.2 Labeled Transition Systems

2.2.1 General Definitions

Just like KS, we need a formal definition of what a Labeled Transition System is.

Definition 7 (Labeled Transition System)**.** A Labeled Transition System T is a 4-tuple $\langle S, I, Act, \rightarrow \rangle$, where:

- S is a set of states.
- I is the set of initial states: $I \subseteq S$.
- Act is a set of actions.
- $\rightarrow$ is a total transition relation: $\rightarrow \subseteq S \times (Act \cup \{\tau\}) \times S$.

Note that from now on, the shorthand notation $s \xrightarrow{a} t$ is used for stating that $(s, a, t) \in \rightarrow$.

Definition 8 (size)**.** The *size* $|T|$ of a Labeled Transition System $T = \langle S, I, Act, \rightarrow \rangle$ is defined as the number of states it has: $|S|$. This means that, given another LTS $T' = \langle S', I', Act', \rightarrow' \rangle$, we say that $|T| \leq |T'|$ (T is *smaller* than T'), if and only if $|S| \leq |S'|$.

2.2.2 Similarity

The notion of simulation equivalence for LTS is very much alike that for KS. In the following definition shows that there is a comparable approach, but in LTS the action of a transition plays the role of the label of a state in KS.

Definition 9 (Simulation relation)**.** Let $T = \langle S, I, Act, \rightarrow \rangle$ be an LTS. A relation $B \subseteq S \times S$ is a simulation relation if and only if for every $s, s' \in S$ such that $(s, s') \in B$:

- For all $t \in S$ and $a \in Act \cup \{\tau\}$, if $s \xrightarrow{a} t$, then $s' \xrightarrow{a} t'$ for some $t' \in S$ such that $(t, t') \in B$.

Definition 10 (Similarity)**.** States $s, s' \in S$ are *similar* (notation: $s \simeq s'$) if and only if there are simulation relations B and B' such that $(s, s') \in B$ and $(s', s) \in B'$.

Definition 11 (Similarity of LTS)**.** Two Labeled Transition Systems T and T' are simulation equivalent ($T \simeq T'$), if and only if:

- For all s in I, there is a s' in I', such that $s \simeq s'$.
- For all s' in I', there is a s in I, such that $s' \simeq s$.

2.2.3 Minimization modulo similarity

In order to follow the approach of KS minimization, the $\forall$-quotient definition for KS has been adapted for LTS. This results in a very similar way of minimizing LTS, using the definitions below.

Using the notion of simulation equivalence between states, we can define the *equivalence classes* C of an LTS T. For every maximal set of similar states $s_1, s_2, \ldots, s_n \in S$ (so, $s_1 \simeq s_2 \simeq \cdots \simeq s_n$), there is one corresponding class $c \in C$. We use "[ ]"to denote the class of a state, so if $s \in c$, then $[s] = c$.

Definition 12 ($\forall$-quotient)**.** We can minimize an LTS T modulo similarity by calculating its $\forall$-quotient system $T_{/\simeq} = \langle S_{/\simeq}, I_{/\simeq}, Act_{/\simeq}, \rightarrow_{/\simeq}\rangle$:

- $S_{/\simeq} = C$, the set of equivalence classes of T
- $I_{/\simeq} = \{c \in C \mid \exists s \in c \; . \; s \in I\}$
- $Act_{/\simeq} = Act$
- $\rightarrow_{/\simeq} = \{(c1, a, c2) \in C \times Act \times C \mid \forall s1 \in c1 \; \exists s2 \in c2 \; . \; (s1, a, s2) \in \rightarrow\}$

The following statements hold for $T_{/\simeq}$:

- $T \simeq T_{/\simeq}$
- For every LTS T' such that $T \simeq T'$, we have that $|T_{/\simeq}| \leq |T'|$.

2.3 Embeddings

When we intend to reuse minimization techniques from one domain to achieve minimization in the other, we need translations in both directions.

In [DNV90], De Nicola and Vaandrager have presented the embeddings lts and ks, allowing us to move from one domain to the other. However, using these translations to go back to the original domain will not result in the original model. For instance, when we start with some Kripke Structure, and first apply embedding lts and then ks, the resulting KS will be much larger than the original one. Reniers and Willemse have dealt with this problem in [RW10] by introducing the reverse embeddings lts^{-1} and ks^{-1}. All of these embeddings are presented here again, slightly changed to allow for initial states.

2.3.1 embedding KS into LTS and back

Definition 13 (The embedding lts)**.** The embedding lts : $KS \rightarrow LTS$ is defined as in [DNV90]: $\text{lts}(K) = \langle S', I', Act, \rightarrow' \rangle$ for an arbitrary Kripke Structure $K = \langle S, I, AP, \rightarrow, L \rangle$, where:

- $S' = S \cup \{\overline{s} \mid s \in S\}$, assuming that $\overline{s} \notin S$ for all $s \in S$.
- $I' = I$
- $Act = 2^{AP} \cup \{\bot\}$
- $\rightarrow'$ is the smallest relation satisfying:

$$\frac{}{s \xrightarrow{\bot}{}' \overline{s}} \qquad \frac{s \rightarrow t \quad L(s) = L(t)}{s \xrightarrow{\tau}{}' t}$$

$$\frac{}{\overline{s} \xrightarrow{L(s)}{}' s} \qquad \frac{s \rightarrow t \quad L(s) \neq L(t)}{s \xrightarrow{L(t)}{}' t}$$

Definition 14 (Reversibility)**.** An LTS $T = \langle S, I, Act, \rightarrow \rangle$ is reversible iff:

- $Act = 2^{AP} \cup \{\bot\}$ for some set AP.
- For all $s \in I$, we have that $s \xrightarrow{\bot}$.
- For all $s, s', s'' \in S$ such that $s \xrightarrow{\bot} s'$ and $s \xrightarrow{\bot} s''$, we have that $(s' \xrightarrow{a} \wedge s'' \xrightarrow{a'}\ ) \Rightarrow a = a'$, for all actions $a, a' \in Act$.

Definition 15 (The reverse embedding lts^{-1})**.** Let $T = \langle S, I, Act, \rightarrow \rangle$ be a reversible LTS. The Kripke Structure $\text{lts}^{-1}(T)$ is the structure $\langle S', I', AP, \rightarrow', L \rangle$, such that:

- $S' = \{s \in S \mid s \xrightarrow{\bot}\}$
- $I' = I$
- AP is such that $Act = 2^{AP} \cup \{\bot\}$
- $\rightarrow'$ is the least relation satisfying:

$$\frac{s \xrightarrow{a} s' \quad a \neq \bot \quad s \xrightarrow{\bot}}{s \rightarrow' s'}$$

- $L(s) = a$ for the unique a such that $s \xrightarrow{\bot} s' \xrightarrow{a} s$ for each $s \in S'$

2.3.2 Embedding LTS into KS and back

Definition 16 (The embedding ks)**.** The embedding ks : $LTS \to KS$ is defined as in [DNV90]: $\mathrm{ks}(T) = \langle S', I', AP, \to', L\rangle$ for an arbitrary LTS $T = \langle S, I, Act, \to\rangle$, where:

- $S' = S \cup \{(s,a,t) \in \to \ \mid a \neq \tau\}$
- $I' = I$
- $AP = Act \cup \{\bot\}$, where $\bot \notin Act$
- $\to'$ is the smallest relation satisfying:

$$\frac{}{s \to' (s,a,t)} \qquad \frac{}{(s,a,t) \to' t} \qquad \frac{s \xrightarrow{\tau} t}{s \to' t}$$

- $L(s) = \{\bot\}$ for $s \in S$, and $L((s,a,t)) = \{a\}$.

Definition 17 (Reversibility)**.** A Kripke Structure $K = \langle S, I, AP, \to, L\rangle$ is reversible iff:

- $AP = Act \cup \{\bot\}$ for some set Act.
- For all $s \in I$, we have that $L(s) = \{\bot\}$.
- $|L(s)| = 1$ for all $s \in S$.
- For all $s \in S$ such that $\bot \notin L(s)$, we have that $\forall s', s'' \in S \ . \ (s \to s' \wedge s \to s'') \Rightarrow (s' = s'' \wedge L(s') = \{\bot\})$

Definition 18 (The reverse embedding ks^{-1})**.** Let $K = \langle S, I, AP, \to, L\rangle$ be a reversible KS. Then $\mathrm{ks}^{-1}(K)$ is the LTS $\langle S', I', Act, \to'\rangle$, such that:

- $S' = \{s \in S | L(s) = \{\bot\}\}$
- $I' = I$
- $Act = AP \setminus \{\bot\}$
- $\to'$ is the least relation satisfying:

$$\frac{s \to s' \quad L(s) = L(s')}{s \xrightarrow{\tau}' s'} \qquad \frac{s \to s'' \quad a \in L(s'') \setminus \{\bot\} \quad s'' \to s'}{s \xrightarrow{a}' s'}$$

3 Combining Minimizations and Embeddings: Similarity

This section shows, using the definitions from the previous section, that it is possible to perform a minimization modulo similarity in either KS or LTS, while using an implementation of minimization in the other.

For notational purposes, the functions $\min_{LTS}$ and $\min_{KS}$ are introduced, such that $\min_{LTS}(T) = T_{/\simeq}$ and $\min_{KS}(K) = K_{/\simeq}$, for arbitrary LTS T and KS K.

3.1 Minimization modulo similarity of KS using minimization in LTS

Lemma 1. *Given a reversible Labeled Transition System T, we have that $T_{/\simeq}$ is reversible.*

Proof. Suppose we have a reversible Labeled Transition System $T = \langle S, I, Act, \rightarrow\rangle$, and its $\forall$-quotient system $T_{/\simeq} = \langle S_{/\simeq}, I_{/\simeq}, Act_{/\simeq}, \rightarrow_{/\simeq}\rangle$. In order to derive a contradiction, let us assume that $T_{/\simeq}$ is not reversible. This means that at least one of the following statements holds:

- $Act_{/\simeq} \neq 2^{AP} \cup \{\bot\}$ for some set AP. As T is reversible, there is some set AP' such that $Act = 2^{AP'} \cup \{\bot\}$. By definition, $Act_{/\simeq} = Act$, so it also holds that $Act_{/\simeq} = 2^{AP'} \cup \{\bot\}$.

- There is an $s \in I_{/\simeq}$, such that $s \stackrel{\bot}{\not\rightarrow}$. From the definition of $\forall$-quotient, it follows that s is an equivalence class of states $s_1, \ldots, s_n \in S$, such that $s_i \in I$, for some $1 \leq i \leq n$. As T is reversible, it holds that $s_i \xrightarrow{\bot}$. Note that $s_1, \ldots, s_n$ are all in the same equivalence class, so $s_j \xrightarrow{\bot} t_j$, for all $1 \leq j \leq n$ and some state t_j. Furthermore, for each pair $s_k, s_m \in s_1, \ldots, s_n$, there are simulation relations B and B' such that $(s_k, s_m) \in B$ and $(s_m, s_k) \in B'$. Then we also have that $(t_k, t_m) \in B$ and $(t_m, t_k) \in B'$, so $t_k \simeq t_m$. This means that there is an equivalence class t of (not necessarily distinct) states $t_1, \ldots, t_n$, such that for all states in class s, there is a $\bot$- transition to a state in class t. So by definition of $\forall$-quotient, $s \xrightarrow{\bot} t$ in $T_{/\simeq}$.

- There exist $s, s', s'' \in S_{/\simeq}$ such that $s \xrightarrow{\bot} s'$, $s \xrightarrow{\bot} s''$, $s' \xrightarrow{a}$ and $s'' \xrightarrow{a'}$, but $a \neq a'$. In this case, s, s' and s'' are equivalence classes of states $s_1, \ldots, s_n \in S$, $s'_1, \ldots, s'_{n'} \in S$ and $s''_1, \ldots, s''_{n''} \in S$ respectively. For every $1 \leq i \leq n$, there are $1 \leq i' \leq n'$ and $1 \leq i'' \leq n''$, with $s_i \xrightarrow{\bot} s_{i'}$ and $s_i \xrightarrow{\bot} s_{i''}$. Furthermore, there are equivalence classes c_1 and c_2 of the states in S, such that for all $1 \leq i' \leq n'$ and $1 \leq i'' \leq n''$, there are $t \in c_1$ and $u \in c_2$ with $s'_{i'} \xrightarrow{a} t$ and $s''_{i''} \xrightarrow{a'} u$ (where $a \neq a'$). So assume an arbitrary s_i in the equivalence class of s. It has $s_i \xrightarrow{\bot} s_{i'}$ and $s_i \xrightarrow{\bot} s_{i''}$ for some $s_{i'}, s_{i''} \in S$, such that $s_{i'} \xrightarrow{a}$ and $s_{i''} \xrightarrow{a'}$, but $a \neq a'$. From this, it follows that T is not reversible, which leads to a contradiction.

Lemma 2. *We have that* $\min_{LTS} \circ \operatorname{lts} \circ \min_{KS} = \operatorname{lts} \circ \min_{KS}$

Proof. Given a Kripke Structure $K = \langle S, I, AP, \rightarrow, L\rangle$ that is minimal with respect to similarity (in KS), we have to show that $\operatorname{lts}(K) = \langle S', I', Act, \rightarrow'\rangle$ is minimal w.r.t. similarity (in LTS), so $\operatorname{lts}(K) = \operatorname{lts}(K)_{/\simeq} = \langle S_{/\simeq}, I_{/\simeq}, Act_{/\simeq}, \rightarrow_{/\simeq}\rangle$. In order to prove that $S' = S_{/\simeq}$, we show that the identity relation on the states of $\operatorname{lts}(K)$ is a similarity relation, and that this relation is maximal.

As K is minimal, we know that the identity relation on S is a maximal similarity relation. From this it follows that the identity relation on S' is also a similarity relation.

Now, in order to derive a contradiction, assume that the identity relation on S' is not a maximal similarity relation, so there are states $s, t \in S'$, such that $s \neq t$ and $s \simeq t$. Four cases are distinguished:

- $s \in S$ and $t \notin S$. By definition of lts, $s \xrightarrow{\perp}$ but $t \not\xrightarrow{\perp}$ in lts(K). Thus, $s \not\simeq t$.
- $s \notin S$ and $t \in S$. By definition of lts, $t \xrightarrow{\perp}$ but $s \not\xrightarrow{\perp}$ in lts(K). Thus, $s \not\simeq t$.
- $s \in S$ and $t \in S$. In K, s and t are not similar. This means that at least one of the following statements holds:
  - There is no simulation relation B with $(s, t) \in B$. Then, assuming $s_0 = s \wedge t_0 = t$, there exist sequences of transitions $s_0 \to s_1 \to \cdots \to s_n$ and $t_0 \to t_1 \to \cdots \to t_{n-1}$ with $L(s_i) = L(t_i)$ (for $i < n$) and there is no state t_n, such that $t_{n-1} \to t_n$ and $L(t_n) = L(s_n)$. This means that in lts(K), the transition $s_{n-1} \xrightarrow{L(s_n)} s_n$ cannot be mimicked from t_{n-1}. Thus, $s \not\simeq t$.
  - There is no simulation relation B with $(t, s) \in B$. The derivation of this contradiction is symmetrical to the previous case, so $s \not\simeq t$.
- $s \notin S$ and $t \notin S$. By definition, there are distinct $s', t' \in S$ such that $s = \overline{s'}$ and $t = \overline{t'}$. The only transitions of s and t are $s \xrightarrow{L(s')} s'$ and $t \xrightarrow{L(t')} t'$. By definition of $s \simeq t$, there are simulation relations B and B' such that $(s, t) \in B$ and $(t, s) \in B'$. This means that also $(s', t') \in B$ and $(t', s') \in B'$. So $s' \simeq t'$ for $s', t' \in S$, which led to a contradiction in the previous item. Concludingly, $s \not\simeq t$.

From this case distinction it follows that $S' = S_{/\simeq}$. This means that by definition of $\forall$-quotient, we also have that $I' = I_{/\simeq}$. Trivially, it holds that $Act = Act_{/\simeq}$. The last step in this proof is to show that $\to' \ = \ \to_{/\simeq}$. Both of the following statements must hold:

- If $s \xrightarrow{a} t$ in lts(K), then also $s \xrightarrow{a} t$ in lts$(K)_{/\simeq}$. As $S' = S_{/\simeq}$, both s and t are the only states of their equivalence classes in lts(K), $c1$ and $c2$ respectively. So there is an a-transition from every state in $c1$ to a state in $c2$. Following from the definition of $\forall$-quotient, this means that $s \xrightarrow{a} t$ in lts$(K)_{/\simeq}$.
- If $s \xrightarrow{a} t$ in lts$(K)_{/\simeq}$, then also $s \xrightarrow{a} t$ in lts(K). Again, we call the equivalence classes of s and t in lts(K), $c1$ and $c2$ respectively. As $s \xrightarrow{a} t$ in lts$(K)_{/\simeq}$, there is an a-transition from every state in $c1$ to a state in $c2$. As s and t are the only states in their equivalence class, this means that $s \xrightarrow{a} t$ in lts(K).

As this shows that also $\to' \ = \ \to_{/\simeq}$, we can conclude that lts$(K) = $ lts$(K)_{/\simeq}$.

Lemma 3. lts^{-1} *is the functional reverse of* lts, *so* $\text{lts}^{-1} \circ \text{lts} = \text{Id}$.

Proof. This proof follows directly from the definitions of lts and lts^{-1} and is given in [RW10].

Lemma 4. $\min_{KS} = \text{lts}^{-1} \circ \min_{LTS} \circ \text{lts} \circ \min_{KS}$ *implies* $\min_{KS} = \text{lts}^{-1} \circ \min_{LTS} \circ \text{lts}$.

Proof. This proof has been given for minimization modulo bisimilarity and stuttering equivalence in [RW10], along with a proof of the preservation and reflection of $\simeq$ by lts. Combining these two elements easily yields the proof for this lemma.

Theorem 5. $\min_{KS} = \text{lts}^{-1} \circ \min_{LTS} \circ \text{lts}$

Proof. It is shown in Lemma 2 that:

$$\min_{LTS} \circ \text{lts} \circ \min_{KS} = \text{lts} \circ \min_{KS}$$

From Lemma 1, it follows that $\min_{LTS} \circ \text{lts} \circ \min_{KS}(K)$ is reversible for an arbitrary Kripke Structure K. This means that we also have:

$$\text{lts}^{-1} \circ \min_{LTS} \circ \text{lts} \circ \min_{KS} = \text{lts}^{-1} \circ \text{lts} \circ \min_{KS}$$

From Lemma 3 it follows that:

$$\text{lts}^{-1} \circ \min_{LTS} \circ \text{lts} \circ \min_{KS} = \min_{KS}$$

Finally, Lemma 4 can be applied to conclude:

$$\min_{KS} = \text{lts}^{-1} \circ \min_{LTS} \circ \text{lts}$$

3.2 Minimization modulo similarity of LTS using minimization in KS

In order to prove that minimization of an LTS can be done using an implementation of minimization in KS, we first need to define an alternative embedding, ks$'$. This embedding is defined as follows:

Definition 19 (The embedding ks$'$)**.** The embedding $\text{ks}' : LTS \to KS$ is defined as $\text{ks}'(T) = \langle S', I', AP, \to', L\rangle$ for an arbitrary LTS $T = \langle S, I, Act, \to\rangle$, where:

- $S' = S \cup \{(a, t) \mid \text{for each } (s, a, t) \in \to \text{ such that } a \neq \tau\}$
- $I' = I$
- $AP = Act \cup \{\bot\}$, where $\bot \notin Act$
- $\to'$ is the smallest relation satisfying:

$$\frac{s \xrightarrow{a} t \quad a \neq \tau}{s \to' (a, t)} \qquad \frac{}{(a, t) \to' t} \qquad \frac{s \xrightarrow{\tau} t}{s \to' t}$$

- $L(s) = \{\bot\}$ for $s \in S$, and $L((a, t)) = \{a\}$.

Lemma 6. $\min_{KS} \circ \text{ks}' = \min_{KS} \circ \text{ks}$

Proof. We want to show that for an arbitrary Labeled Transition System T, it holds that $\min_{KS} \circ \text{ks}'(T)$ is isomorphic to $\min_{KS} \circ \text{ks}(T)$. In order to derive a contradiction, let us assume that there exists an LTS $T = \langle S, I, Act, \to\rangle$ such that $\min_{KS} \circ \text{ks}'(T) \neq \min_{KS} \circ \text{ks}(T)$. We have that $\min_{KS} \circ \text{ks}'(T) = \langle S', I', AP', \to', L'\rangle$ and $\min_{KS} \circ \text{ks}(K) = \langle S'', I'', AP'', \to'', L''\rangle$. As we assumed that $\min_{KS} \circ \text{ks}'(T) \neq \min_{KS} \circ \text{ks}(T)$, one of the following statements must hold:

- $S' \neq S''$. The only difference in the definitions of ks and ks$'$ is that for every transition in $\rightarrow$, there is a state (s, a, t) in ks(T), while there is a state (a, t) in ks$'(T)$. This means that if there are multiple states s such that $s \xrightarrow{a} t$ (for some action a and some state t), there will be only one state (a, t) in ks$'(T)$. In ks(T) however, there will be a state (s, a, t) for each s. As all of these states (s, a, t) have the same label (a), and their only outgoing transition is to t, it is easy to see that they are all simulation equivalent. Thus, they will be grouped together in an equivalence class by definition of $\forall$-quotient. Concludingly, the minimization eliminates the redundant 'transition'-states (s, a, t) in ks(T), such that $S' = S''$.
- $I' \neq I''$. From the definitions of ks and ks$'$, it follows that the set of initial states of both ks$'(T)$ and ks(T) is I. As the previous case showed that $S' = S''$, we can conclude from the definition of $\forall$-quotient that $I' = I''$.
- $AP' \neq AP''$. It trivially holds by definition of ks, ks$'$ and $\forall$-quotient that $AP' = AP''$.
- $\rightarrow' \neq \rightarrow''$. As we discussed in the first case, the difference between ks and ks$'$ is that the 'transition'-states (s, a, t) of ks for all s with $s \xrightarrow{a} t$, are grouped together into a single state (a, t) in ks$'$. We also showed already that all these states (s, a, t) will be in the same equivalence class c when minimizing. There will be an outgoing transition from c to the equivalence class of t, just like there will be an outgoing transition from (a, t) to the equivalence class of t in ks$'$. Moreover, for all equivalence classes d of the states s with $s \xrightarrow{a} t$, there will be a transition from d to c in ks if and only if there is a transition from d to c in ks$'$. As ks and ks$'$ treat the transitions, the τ-transitions, in exactly the same way, we can conclude that $\rightarrow' = \rightarrow''$.
- $L' \neq L''$. Using the first case, where we showed that $S' = S''$, it is straightforward to see that $L' = L''$.

As this case distinction shows that there exists no LTS T such that $\min_{KS} \circ \text{ks}'(T) \neq \min_{KS} \circ \text{ks}(T)$, we can conclude that $\min_{KS} \circ \text{ks}' = \min_{KS} \circ \text{ks}$.

Lemma 7. *We have that* $\min_{KS} \circ \text{ks}' \circ \min_{LTS} = \text{ks}' \circ \min_{LTS}$

Proof. Given a Labeled Transition System $T = \langle S, I, Act, \rightarrow \rangle$ that is minimal with respect to similarity (in LTS), we have to show that $\text{ks}'(T) = \langle S', I', AP, \rightarrow', L \rangle$ is minimal w.r.t. similarity (in KS), so $\text{ks}'(T) = \text{ks}'(T)_{/\simeq} = \langle S_{/\simeq}, I_{/\simeq}, AP_{/\simeq}, \rightarrow_{/\simeq}, L_{/\simeq} \rangle$. In order to prove that $S' = S_{/\simeq}$, we show that the identity relation on the states of ks$'(T)$ is a similarity relation, and that this relation is maximal.

As T is minimal, we know that the identity relation on S is a maximal similarity relation. From this it follows that the identity relation on S' is also a similarity relation.

Now, in order to derive a contradiction, assume that the identity relation on S' is not a maximal similarity relation, so there are states $s, t \in S'$, such that $s \neq t$ and $s \simeq t$. Four cases are distinguished:

- $s \in S$ and $t \notin S$. By definition of ks$'$, $L(s) = \{\bot\}$ but $L(t) \neq \{\bot\}$ in ks$'(T)$. Thus, $s \not\simeq t$.
- $s \notin S$ and $t \in S$. By definition of ks$'$, $L(t) = \{\bot\}$ but $L(s) \neq \{\bot\}$ in ks$'(T)$. Thus, $s \not\simeq t$.

- $s \in S$ and $t \in S$. In T, s and t are not similar. This means that at least one of the following statements holds:
  - There is no simulation relation B with $(s,t) \in B$. Then, assuming $s_0 = s \wedge t_0 = t$, there exist sequences of transitions $s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} s_n$ and $t_0 \xrightarrow{a_1} t_1 \xrightarrow{a_2} \dots \xrightarrow{a_{n-1}} t_{n-1}$ and there is no state t_n, such that $t_{n-1} \xrightarrow{a_n} t_n$. This means that in $\mathrm{ks}'(T)$, the transition $s_{n-1} \to (a_n, s_n)$ cannot be mimicked from t_{n-1}. Thus, $s \not\simeq t$.
  - There is no simulation relation B with $(t,s) \in B$. The derivation of this contradiction is symmetrical to the previous case, so $s \not\simeq t$.
- $s \notin S$ and $t \notin S$. By definition of ks', there are $(s', a, s''), (t', a', t'') \in \to$ such that $s = (a, s'')$ and $t = (a', t'')$. As $s \neq t$, one of the following statements holds:
  - $a \neq a'$. Then $L(s) \neq L(t)$, so $s \not\simeq t$.
  - $s'' \neq t''$. Then there is a transition $s \to s''$ in $\mathrm{ks}'(T)$, which cannot be mimicked from t, as the only transition from t is $t \to t''$ and $s'' \not\simeq t''$. So, $s \not\simeq t$.

From this case distinction it follows that $S' = S_{/\simeq}$. This means that by definition of $\forall$-quotient, we also have that $I' = I_{/\simeq}$ and $L' = L_{/\simeq}$. Trivially, it holds that $AP = AP_{/\simeq}$. The last step in this proof is to show that $\to' \;=\; \to_{/\simeq}$. Both of the following statements must hold:

- If $s \to t$ in $\mathrm{ks}'(T)$, then also $s \to t$ in $\mathrm{ks}'(T)_{/\simeq}$. As $S' = S_{/\simeq}$, both s and t are the only states of their equivalence classes in $\mathrm{ks}'(T)$, $c1$ and $c2$ respectively. So there is an transition from every state in $c1$ to a state in $c2$. Following from the definition of $\forall$-quotient, this means that $s \to t$ in $\mathrm{ks}'(T)_{/\simeq}$.
- If $s \to t$ in $\mathrm{ks}'(T)_{/\simeq}$, then also $s \to t$ in $\mathrm{ks}'(T)$. Again, we call the equivalence classes of s and t in $\mathrm{ks}'(T)$, $c1$ and $c2$ respectively. As $s \to t$ in $\mathrm{ks}'(T)_{/\simeq}$, there is an transition from every state in $c1$ to a state in $c2$. As s and t are the only states in their equivalence class, this means that $s \to t$ in $\mathrm{ks}'(T)$.

As this shows that also $\to' \;=\; \to_{/\simeq}$, we can conclude that $\mathrm{ks}'(T) = \mathrm{ks}'(T)_{/\simeq}$.

Lemma 8. *Given a reversible Kripke Structure K, we have that $K_{/\simeq}$ is reversible.*

Proof. Suppose we have a reversible Kripke Structure $K = \langle S, I, AP, \to, L\rangle$, and its $\forall$-quotient system $K_{/\simeq} = \langle S_{/\simeq}, I_{/\simeq}, AP_{/\simeq}, \to_{/\simeq}, L_{/\simeq}\rangle$. In order to derive a contradiction, let us assume that $K_{/\simeq}$ is not reversible. This means that at least one of the following statements holds:

- $AP_{/\simeq} \neq Act \cup \{\perp\}$ for some set Act. As K is reversible, there is some set Act' such that $AP = Act' \cup \{\perp\}$. By definition, $AP_{/\simeq} = AP$, so it also holds that $AP_{/\simeq} = Act' \cup \{\perp\}$.
- There is an $s \in I_{/\simeq}$, such that $L(s) \neq \{\perp\}$. From the definition of $\forall$-quotient, it follows that s is an equivalence class of states $s_1, \dots, s_n \in S$, such that $s_i \in I$, for some $1 \leq i \leq n$. As T is reversible, it holds that $L(s_i) = \{\perp\}$. Note that $s_1, \dots, s_n$ are all in the same equivalence class, so $L(s_j) = \{\perp\}$, for all $1 \leq j \leq n$. By definition, this also means that $L(s) = \{\perp\}$.

- There is an $s \in S_{/\simeq}$, such that $|L(s)| \neq 1$. From the definitions of $\forall$-quotient and similarity, it follows that s is an equivalence class of states $s_1, \ldots, s_n \in S$, such that $L(s_i) = L(s_j)$ for all $1 \leq i \leq j \leq n$. As K is reversible, $|L(s_i)| = 1$ for every $1 \leq i \leq n$. By definition, $L(s) = L(s_i)$ (again for $1 \leq i \leq n$, so $|L(s)| = 1$.
- There exist $s, s', s'' \in S_{/\simeq}$, with $\bot \notin L(s)$, $s \to s'$, $s \to s''$, such that $s' \neq s'' \vee L(s') \neq \{\bot\}$. In this case, s, s' and s'' are equivalence classes of states $s_1, \ldots, s_n \in S$, $s'_1, \ldots, s'_{n'} \in S$ and $s''_1, \ldots, s''_{n''} \in S$ respectively. For every $1 \leq i \leq n$, there are $1 \leq i' \leq n'$ and $1 \leq i'' \leq n''$, with $s_i \to s'_{i'}$ and $s_i \to s''_{i''}$. Now at least one of the following statements is true:
  - $s' \neq s''$, so $\{s'_1, \ldots, s'_{n'}\} \neq \{s''_1, \ldots, s''_{n''}\}$. As equivalence classes are disjunct, we have that (for every $1 \leq i \leq n$) there are $s'_{i'}$ and $s''_{i''}$, such that $\bot \notin L(s_i)$, $s_i \to s'_{i'}$, $s_i \to s''_{i''}$ and $s'_{i'} \neq s''_{i''}$. This contradicts the assumption that K is reversible, so $s' = s''$.
  - $L(s') \neq \{\bot\}$, so $L(s'_{i'}) \neq \{\bot\}$ (for all $1 \leq i' \leq n$). Then there exists s_i with $\bot \notin L(s_i)$ and $s_i \to s'_{i'}$, such that $L(s'_{i'}) \neq \{\bot\}$. This contradicts the assumption that K is reversible, so $L(s') = \{\bot\}$.

Lemma 9. ks^{-1} *is the functional reverse of* ks', *so* $\mathrm{ks}^{-1} \circ \mathrm{ks}' = \mathrm{Id}$.

Proof. This proof follows directly from the definitions of ks' and ks^{-1} and very similar to the proof for ks and ks^{-1} as given in [RW10].

Lemma 10. $\min_{LTS} = \mathrm{ks}^{-1} \circ \min_{KS} \circ \mathrm{ks}' \circ \min_{LTS}$ *implies* $\min_{LTS} = \mathrm{ks}^{-1} \circ \min_{KS} \circ \mathrm{ks}'$.

Proof. This proof is symmetrical to that of Lemma 4.

Lemma 11. $\min_{LTS} = \mathrm{ks}^{-1} \circ \min_{KS} \circ \mathrm{ks}'$

Proof. This proof is symmetrical to that of Theorem 5, using Lemmas 7-10 instead of Lemmas 1-4.

Theorem 12. $\min_{LTS} = \mathrm{ks}^{-1} \circ \min_{KS} \circ \mathrm{ks}$

Proof. From Lemma 6 we have that $\min_{KS} \circ \mathrm{ks}' = \min_{KS} \circ \mathrm{ks}$, meaning that for an arbitrary LTS T it holds that $\min_{KS} \circ \mathrm{ks}'(T)$ is isomorphic to $\min_{KS} \circ \mathrm{ks}(T)$. This means that we can easily conclude from Lemma 11 that $\min_{LTS} = \mathrm{ks}^{-1} \circ \min_{KS} \circ \mathrm{ks}$.

4 An Example of Minimization modulo Similarity

This section gives an example of minimization modulo similarity of a Kripke Structure, while using minimization techniques for Labeled Transition Systems. The intention is to give a visualisation of Theorem 5, in order to give a better view on how the embeddings and minimizations work in practice.

First, we present the example Kripke Structure K:

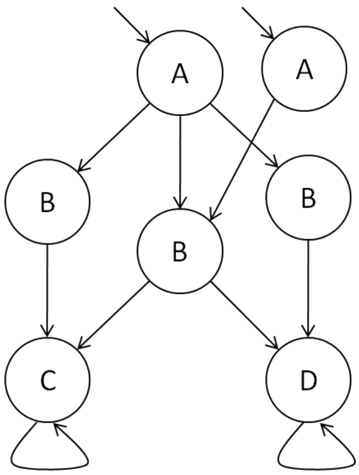

Figure 1: The original Kripke Structure K.

Minimizing K using the definition of $\forall$-quotient yields:

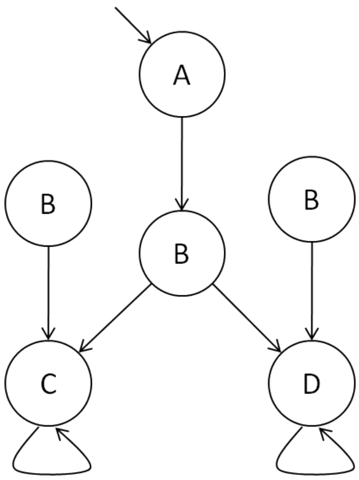

Figure 2: $\min_{KS}(K)$, the $\forall$-quotient of K.

Translating K into an LTS using the embedding lts results in the following structure:

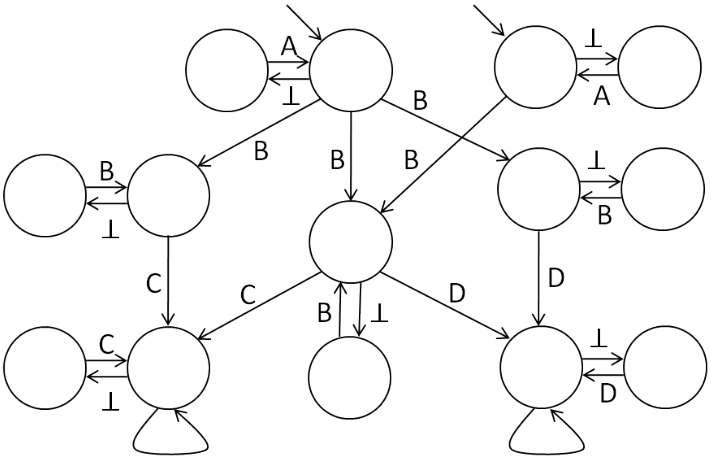

Figure 3: The embedding lts(K).

Applying the ∀-quotient definition on lts(K) in the LTS domain yields:

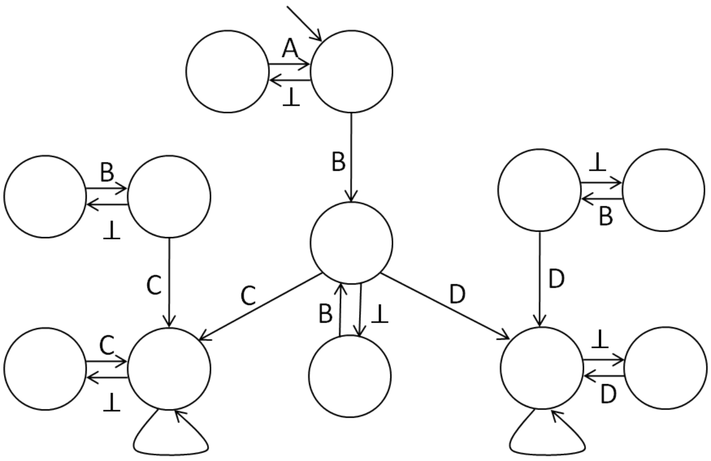

Figure 4: The minimized embedding $\min_{LTS} \circ \, \text{lts}(K)$.

When we apply the reverse embedding lts^{-1} to translate $\min_{LTS} \circ \text{lts}(K)$ back to a Kripke Structure gives us the final result:

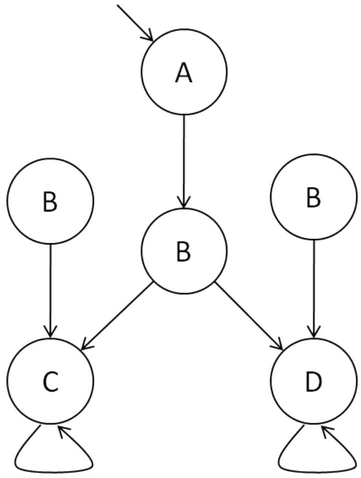

Figure 5: The reversed minimized embedding $\text{lts}^{-1} \circ \min_{LTS} \circ \text{lts}(K)$.

Note that in Figures 2 and 5 we see that for the example Kripke Structure K, it holds that $\min_{KS}(K) = \text{lts}^{-1} \circ \min_{LTS} \circ \text{lts}(K)$.

5 Alternative embeddings from KS to LTS

The embedding lts as presented in section 2 is the most common translation from KS to LTS in literature. This observation contributed to chosing this embedding as our main embedding for the proofs and examples in this document. However, applying the embedding results in an LTS that is a lot larger than the original KS (it has twice as many states), and the behaviour of such an LTS is intuitively not equivalent to the behaviour of the KS. More specifically, an execution of the LTS could loop infinitely within a state s and its dual $\overline{s}$.

Therefore, we have reasons to look for 'better' embeddings. Such an embedding should provide us with an LTS that is more intuitively related to the original KS, while preserving the properties that [DNV90] and [RW10] have proved for lts. Many possible alternatives have been suggested, including the following:

Definition 20 (The embedding lts$'$)**.** The embedding $\text{lts}' : KS \rightarrow LTS$ is defined as: $\text{lts}'(K) = \langle S', I', Act, \rightarrow' \rangle$ for an arbitrary Kripke Structure $K = \langle S, I, AP, \rightarrow, L \rangle$, where:

- $S' = S \cup \{\overline{s} \mid s \in S\}$, assuming that $\overline{s} \notin S$ for all $s \in S$.
- $I' = I$
- $Act = 2^{AP} \cup \{\bot\}$
- $\rightarrow'$ is the smallest relation satisfying:

$$\frac{}{s \xrightarrow{L(s)}{}' \ \bar{s}} \qquad \frac{s \to t \quad L(s) = L(t)}{\bar{s} \xrightarrow{\tau}{}' \ \bar{t}}$$

$$\frac{s \to t \quad L(s) \neq L(t)}{\bar{s} \xrightarrow{L(t)}{}' \ \bar{t}}$$

While this embedding still has as many states as lts, its behaviour is more obviously related to that of the original KS, as the aforementioned infinite loop within a state and its dual is no longer possible.

This alternative embeddings is only an example for translations that may be an improvement to lts, but it shows how a subtle change can give an entirely different result. In this document, we do not focus on finding the best possible embedding, but a future research may be required to investigate a more satisfying alternative.

6 Conclusions

Extending the works of [DNV90] and [RW10], the interchangeability of KS and LTS for model minimization modulo similarity has been established. In section 3, an extensive proof is given that the embeddings lts and ks can be applied to reuse minimization techniques that maintain similarity.

As this is no trivial property, but shows a rather strong connection between KS and LTS, these results contribute to the accepted assumption that the two languages are equally expressive.

7 Future work

Along with the investigation of minimizing while maintaining bisimilarity and stuttering equivalence by [RW10], and the focus on similarity in this document, the most important equivalence remaining for Kripke Structures is trace equivalence. While some intuitions have been presented, a more formal proof of the interchangability between KS and LTS would practically complete the minimization topic. Hence, this could be an obvious extension.

Of course, there are many more operations besides minimization that may be interesting. In a similar manner to this document, one could investigate whether it is possible to achieve useful results in one domain while using techniques in the other domain.

Finally, there are reasons to look for an improved version of the embedding lts. It produces a rather large LTS, while there may be nicer translations possible. As pointed out in section 5, there are many ways to make a translation from KS to LTS, and it would be useful to find the best possible embedding.

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