The Small Progress Measures algorithm for Parity games

Material: "Small Progress Measures for Solving Parity Games", Marcin Jurdziński

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Parity games

Recall:

Definition (Parity game)

A parity game Γ is a four tuple $(V, E, p, (V_{Even}, V_{Odd}))$, where:

- (V, E) is a directed graph,
- E is a total edge relation,
- $p : V \rightarrow \mathbb{N}$ assigns priorities to vertices, and
- (V_{Even}, V_{Odd}) is a partitioning of V

- A player does a step in the game if a token is on a vertex owned by that player;
- A play (denoted π) is an infinite sequence of steps.

Strategies

- A strategy for $Player$ is a partial function $\psi_{Player}\colon V^* \times V_{Player} \rightarrow V$.
- A play $\pi = v_1 v_2 v_3 \ldots$ is consistent with strategy ψ_{Player} for $Player$ iff every $v_i \in \pi$ such that $v_i \in V_{Player}$ is immediately followed by $v_{i+1} = \psi_{Player}(v_1 \ldots v_i)$.

Definition (Memoryless strategy)

A memoryless strategy for $Player$ is a partial function $\psi_{Player}\colon V_{Player} \rightarrow V$ that decides the vertex the token is played to based on the current vertex.

Winning a parity game

Let $\pi = v_1 v_2 v_3 \ldots$ be a play:

- $\inf(\pi)$ denotes set of priorities occurring infinitely often in π;
- π is winning for player Even iff $\min(\inf(\pi))$ is even;

Definition (Winning strategy)

Strategy ψ_{Player} is a winning strategy for Player from set $W \subseteq V$ if every play starting from a vertex in W, consistent with ψ_{Player} is winning for Player.

- There is a memoryless winning strategy for Player from $W \subseteq V$ iff there is a winning strategy for Player from W.

Goal

Let $G = (V, E, p, (V_{Even}, V_{Odd}))$ be a parity game.

- There is a unique partition (W_{Even}, W_{Odd}) of V such that:
  - $Even$ has winning strategy ψ_{Even} from W_{Even}, and
  - Odd has winning strategy ψ_{Odd} from W_{Odd}.

Goal of parity game algorithms

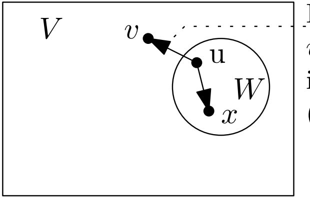
Compute partitioning (W_{Even}, W_{Odd}) with strategies ψ_{Even} and ψ_{Odd} of V, such that ψ_{Even} is winning for player $Even$ from W_{Even} and ψ_{Odd} is winning for player Odd from W_{Odd}.

Closedness and cycles

Let $G = (V, E, p, (V_{Even}, V_{Odd}))$ be a parity game. A strategy ψ_{Even} is closed on a set $W \subseteq V$ if for all $v \in W$, we have:

- if $v \in V_{Even}$ then $\psi_{Even}(v) \in W$, and
- if $v \in V_{Odd}$ then $(v, w) \in E$ implies $w \in W$.

Each play consistent with strategy ψ_{Even} closed on W, starting in W, stays within W.

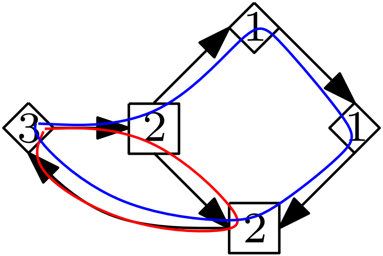


Edges (u, v) only for $u \in V_{Even}$, and only if there also is edge (u, x) for $x \in W$.

Cycles

A cycle in a parity game is a path $v_1, v_2, \ldots, v_n$, with $v_1 = v_n$.
We say that a cycle $v_1, v_2, \ldots, v_n$ is

- an i-cycle if $i = \min\{p(v_j) \mid 1 \leq j \leq n\}$, *i.e.* i is the smallest priority occurring on the cycle.
- an even cycle if it is an i-cycle, i is even.



Characterization of winning strategies

Let $G = (V, E, p, (V_{Even}, V_{Odd}))$ be a parity game. Let ψ_{Even} be a strategy for player $Even$, closed on $W \subseteq V$. Define the game $G' = (V', E', p', (V'_{Even}, V'_{Odd}))$ such that:

- $V' = W$,
- $V'_{Even} = V_{Even} \cap V'$
- $V'_{Odd} = V_{Odd} \cap V'$
- $E' = \{(v, w) \mid v \in V'_{Even} \wedge w = \psi_{Even}(v)\} \cup \{(v, w) \mid (v, w) \in E \wedge v \in V'_{Odd}\}$
- $p'(v) = p(v)$ for $v \in V'$

ψ_{Even} is winning for player $Even$ from W if and only if all cycles in G' are even.

Aim of small progress measures

Aim

Characterize the cycles reachable from each vertex using a measure, such that:

- the measure is computable using fixed point iteration,
- the measure assigned to a vertex contains for all odd priorities the maximal number of times this priority can be seen if player Odd moves over the graph, until a vertex with smaller priority is seen.

Notation

Let $\alpha \in \mathbb{N}^d$ be a d-tuple of non-negative integers.

- we number its components from 0 to $d-1$, i.e. $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{d-1})$,
- $<, \leq, =, \neq, \geq, >$ on tuples denote lexicographic ordering,
- $(n_0, n_1, \ldots, n_k) \equiv_i (m_0, m_1, \ldots, m_l)$ iff $(n_0, n_1, \ldots, n_i) \equiv (m_0, m_1, \ldots, m_i)$, for $\equiv \in \{<, \leq, =, \neq, \geq, >\}$
- Note that if $i > k$ or $i > l$, the tuples may be suffixed with 0s

Example

- $(0,1,0,1) =_0 (0,2,0,1) \equiv (0) = (0) \equiv$ true
- $(0,1,0,1) <_1 (0,2,0,1) \equiv (0,1) < (0,2) \equiv$ true
- $(0,1,0,1) \geq_3 (0,2,0,1) \equiv (0,1,0,1) \geq (0,2,0,1) \equiv$ false

Notation

Let $G = (V, E, p, (V_{Even}, V_{Odd}))$ be a parity game, and let $d = \max\{p(v) \mid v \in V\} + 1$.

- For $i \in \mathbb{N}$, let $V_i = \{v \mid p(v) = i \wedge v \in V\}$,
- denote $n_i =\mid V_i \mid$, the number of vertices with priority i,

Definition ($\mathbb{M}_G$)

Define $\mathbb{M}_G \subseteq \mathbb{N}^d$, such that it is the finite set of d-tuples, with 0 on even positions, and non-negative integers bounded by n_i on odd positions i.

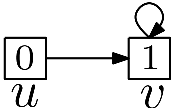
Parity progress measure

Idea: characterize vertices that can only reach even cycles.

Definition (Parity progress measure)

Let $G = (V, E, p, (V_{Even}, V_{Odd}))$ be a parity game. Function $\varrho : V \to \mathbb{N}^d$ is a parity progress measure for G if for all $(v, w) \in E$ it holds that:

- $\varrho(v) \geq_{p(v)} \varrho(w)$ if $p(v)$ is even
- $\varrho(v) >_{p(v)} \varrho(w)$ if $p(v)$ is odd



Problem: no parity progress measure can be assigned to these vertices, as parity progress measure only exists for even cycles. (Second clause requires $\varrho(v) >_1 \varrho(v)$)

Allowing odd cycles

Let $G = (V, E, p, (V_{Even}, V_{Odd}))$ be a parity game.

Define $\mathbb{M}_G^\top = \mathbb{M}_G \cup \{\top\}$, such that:

- $m\{<, <_i\}\top$ for all $m \in \mathbb{M}_G$,
- $\top =_i \top$ for all i.

Definition (Prog)

If $\varrho : V \to \mathbb{M}_G^\top$ and $(v, w) \in E$, then $Prog(\varrho, v, w)$ is the least $m \in \mathbb{M}_G^\top$, such that

- $m \geq_{p(v)} \varrho(w)$ if $p(v)$ is even,
- $m >_{p(v)} \varrho(w)$, or $m = \varrho(w) = \top$ if $p(v)$ is odd.

Example

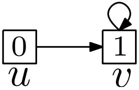
Recall the definition of $Prog$:

Definition (Prog)

If $\varrho{:}V \to \mathbb{M}_G^{\top}$ and $(v, w) \in E$, then $Prog(\varrho, v, w)$ is the least $m \in \mathbb{M}_G^{\top}$, such that

- $m \geq_{p(v)} \varrho(w)$ if $p(v)$ is even,
- $m >_{p(v)} \varrho(w)$, or $m = \varrho(w) = \top$ if $p(v)$ is odd.

$\varrho(u) = \top$



$\varrho(v) = \top$

Measure can identify both $Even$ and Odd reachable cycles.

Game parity progress measure

Definition (Game parity progress measure)

Let $G = (V, E, p, (V_{Even}, V_{Odd}))$ be a parity game. A function $\varrho{:}V \rightarrow \mathbb{M}_G^{\top}$ is a game parity progress measure if for all $v \in V$, it holds that:

- if $v \in V_{Even}$, then $\exists_{(v,w)\in E}\, \varrho(v) \geq_{p(v)} Prog(\varrho, v, w)$;
- if $v \in V_{Odd}$, then $\forall_{(v,w)\in E}\, \varrho(v) \geq_{p(v)} Prog(\varrho, v, w)$, and

Note: if ϱ is a game parity progress measure, then $\varrho(v) \neq \top$ if and only if all cycles reachable from vertex v are even.

Strategies from progress measures

Let $G = (V, E, p, (V_{Even}, V_{Odd}))$ be a parity game, and $\varrho{:}V \to \mathbb{M}_G^\top$ be a game parity progress measure.

- Define strategy $\overline{\varrho}{:}V_{Even} \to V$ for player $Even$, by setting $\overline{\varrho}(v)$ to be a successor w of v that minimizes $\varrho(w)$.
- Let $\| \varrho \| = \{v \mid v \in V \wedge \varrho(v) \neq \top\}$

Properties:

- If ϱ is a game parity progress measure, then $\overline{\varrho}$ is a winning strategy for player $Even$ from $\| \varrho \|$.
- There is a game parity progress measure $\varrho{:}V \to \mathbb{M}_G^\top$ such that $\| \varrho \|$ is the winning set of player $Even$.

Fixed points

Characterize game parity progress measure as fixed point of monotone operators in a finite complete lattice:

- a least game parity progress measure μ exists (Knaster-Tarski),
- computable by fixed point iteration (see Lecture 3, slide 13 for an algorithm),
- $|| \mu ||$ is winning set of player $Even$

Let $G = (V, E, p, (V_{Even}, V_{Odd}))$, and $\mu, \varrho{:}V \to \mathbb{M}_G^\top$.

- $\mu \sqsubseteq \varrho$ if $\mu(v) \leq \varrho(v)$ for all $v \in V$
- write $\mu \sqsubset \varrho$ if $\mu \sqsubseteq \varrho$ and $\mu \neq \varrho$.

$\sqsubseteq$ gives a complete lattice structure on the set of functions $V \to \mathbb{M}_G^\top$.

Lifting progress measures

Define $\mathit{Lift}(\varrho, v)$ for $v \in V$ as follows:

$$\mathit{Lift}(\varrho, v)(u) = \begin{cases} \varrho(u) & \text{if } u \neq v, \\ \min\{\mathit{Prog}(\varrho, v, w) \mid (v, w) \in E\} & \text{if } u = v \in V_{\mathit{Even}} \\ \max\{\mathit{Prog}(\varrho, v, w) \mid (v, w) \in E\} & \text{if } u = v \in V_{\mathit{Odd}} \end{cases}$$

Observe:

- For every $v \in V$, $\mathit{Lift}(\cdot, v)$ is $\sqsubseteq$-monotone.
- A function $\varrho{:}V \to \mathbb{M}_G^{\top}$ is a game parity progress measure if and only if $\mathit{Lift}(\varrho, v) \sqsubseteq \varrho$ for all $v \in V$.

The algorithm

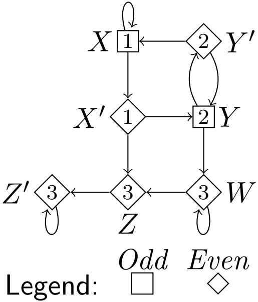
The least game parity progress measure can now be computed using fixed point approximation:

Algorithm *(ProgressMeasureLifting)*

$\mu \leftarrow \lambda v \in V.(0, \ldots, 0)$
while $\mu \sqsubset \mathit{Lift}(\mu, v)$ for some $v \in V$ **do**
    $\mu \leftarrow \mathit{Lift}(\mu, v)$
end while

Example

Consider parity game G:



Legend:

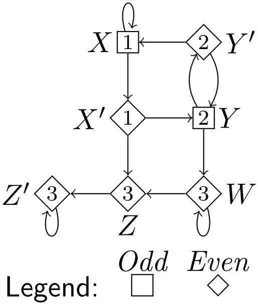
Initially: $\mu \leftarrow \lambda v \in V.(0,0,0,0)$, so

v	$\mu(v)$
X	$(0,0,0,0)$
X'	$(0,0,0,0)$
Y	$(0,0,0,0)$
Y'	$(0,0,0,0)$
Z	$(0,0,0,0)$
Z'	$(0,0,0,0)$
W	$(0,0,0,0)$

$Lift(\mu, X) = \max\{Prog(\mu, X, X'), Prog(\mu, X, X)\} = \max\{(0,1,0,0),(0,1,0,0)\} = (0,1,0,0)$

Example

Consider parity game G:

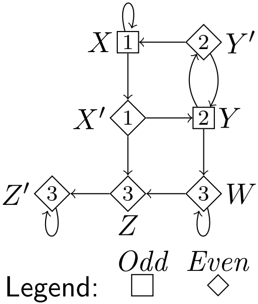


v	$\mu(v)$
X	$(0,1,0,0)$
X'	$(0,0,0,0)$
Y	$(0,0,0,0)$
Y'	$(0,0,0,0)$
Z	$(0,0,0,0)$
Z'	$(0,0,0,0)$
W	$(0,0,0,0)$

$$\mathit{Lift}(\mu, X) = \max\{\mathit{Prog}(\mu, X, X'), \mathit{Prog}(\mu, X, X)\} = \max\{(0,1,0,0),(0,2,0,0)\} = (0,2,0,0)$$

Example

Consider parity game G:

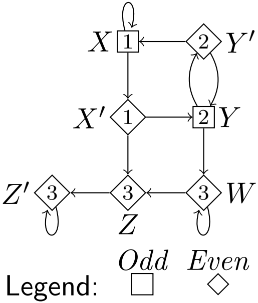


v	$\mu(v)$
X	$(0,2,0,0)$
X'	$(0,0,0,0)$
Y	$(0,0,0,0)$
Y'	$(0,0,0,0)$
Z	$(0,0,0,0)$
Z'	$(0,0,0,0)$
W	$(0,0,0,0)$

$$Lift(\mu, X) = \max\{Prog(\mu, X, X'), Prog(\mu, X, X)\} = \max\{(0,1,0,0), \top\} = \top$$

Example

Consider parity game G:



v	$\mu(v)$
X	$\top$
X'	$(0,0,0,0)$
Y	$(0,0,0,0)$
Y'	$(0,0,0,0)$
Z	$(0,0,0,0)$
Z'	$(0,0,0,0)$
W	$(0,0,0,0)$

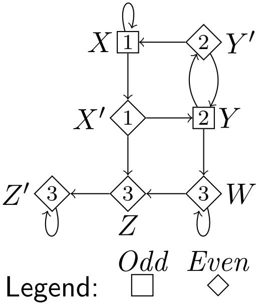
$$Lift(\mu, Y') = \min\{Prog(\mu, Y', X), Prog(\mu, Y', Y)\} = \min\{\top, (0,0,0,0)\} = (0,0,0,0)$$

$$Lift(\mu, Y) = \max\{Prog(\mu, Y, W), Prog(\mu, Y, Y')\} = \max\{(0,0,0,0), (0,0,0,0)\} = (0,0,0,0)$$

$$Lift(\mu, X') = \min\{Prog(\mu, X', Y), Prog(\mu, X', Z)\} = \min\{(0,1,0,0), (0,1,0,0)\} = (0,1,0,0)$$

Example

Consider parity game G:

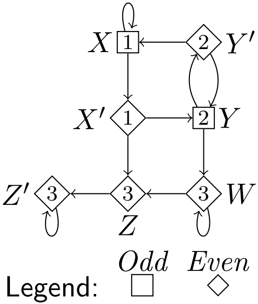


v	$\mu(v)$
X	$\top$
X'	$(0,1,0,0)$
Y	$(0,0,0,0)$
Y'	$(0,0,0,0)$
Z	$(0,0,0,0)$
Z'	$(0,0,0,0)$
W	$(0,0,0,0)$

$Lift(\mu, Z') = \min\{Prog(\mu, Z', Z')\} = \min\{(0,0,0,1)\} = (0,0,0,1)$

Example

Consider parity game G:

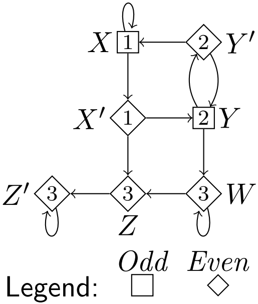


v	$\mu(v)$
X	$\top$
X'	$(0,1,0,0)$
Y	$(0,0,0,0)$
Y'	$(0,0,0,0)$
Z	$(0,0,0,0)$
Z'	$(0,0,0,1)$
W	$(0,0,0,0)$

$Lift(\mu, Z') = \min\{Prog(\mu, Z', Z')\} = \min\{(0,0,0,2)\} = (0,0,0,2)$

Example

Consider parity game G:

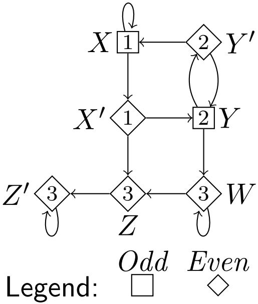


v	$\mu(v)$
X	$\top$
X'	$(0,1,0,0)$
Y	$(0,0,0,0)$
Y'	$(0,0,0,0)$
Z	$(0,0,0,0)$
Z'	$(0,0,0,2)$
W	$(0,0,0,0)$

$Lift(\mu, Z') = \min\{Prog(\mu, Z', Z')\} =$
$\min\{(0,0,0,3)\} = (0,0,0,3)$

Example

Consider parity game G:

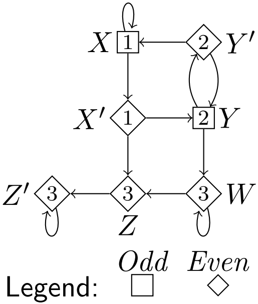


v	$\mu(v)$
X	$\top$
X'	$(0,1,0,0)$
Y	$(0,0,0,0)$
Y'	$(0,0,0,0)$
Z	$(0,0,0,0)$
Z'	$(0,0,0,3)$
W	$(0,0,0,0)$

$$Lift(\mu, Z') = \min\{Prog(\mu, Z', Z')\} = \min\{(0,1,0,0)\} = (0,1,0,0)$$

Example

Consider parity game G:

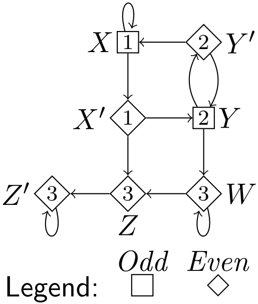


v	$\mu(v)$
X	$\top$
X'	$(0,1,0,0)$
Y	$(0,0,0,0)$
Y'	$(0,0,0,0)$
Z	$(0,0,0,0)$
Z'	$(0,1,0,0)$
W	$(0,0,0,0)$

$Lift(\mu, Z') = \min\{Prog(\mu, Z', Z')\} = \min\{(0,1,0,1)\} = (0,1,0,1)$

Example

Consider parity game G:



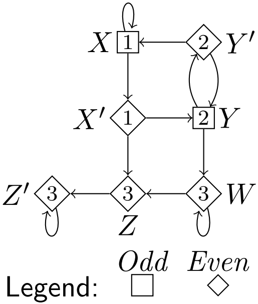
v	$\mu(v)$
X	$\top$
X'	$(0,1,0,0)$
Y	$(0,0,0,0)$
Y'	$(0,0,0,0)$
Z	$(0,0,0,0)$
Z'	$(0,1,0,1)$
W	$(0,0,0,0)$

Repeat lifting Z' even more often

$$Lift(\mu, Z') = \min\{Prog(\mu, Z', Z')\} = \min\{\top\} = \top$$

Example

Consider parity game G:

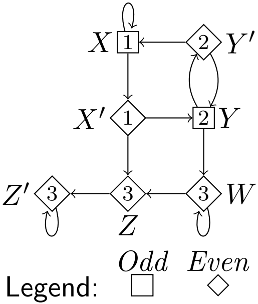


v	$\mu(v)$
X	$\top$
X'	$(0, 1, 0, 0)$
Y	$(0, 0, 0, 0)$
Y'	$(0, 0, 0, 0)$
Z	$(0, 0, 0, 0)$
Z'	$\top$
W	$(0, 0, 0, 0)$

$$Lift(\mu, Z) = \min\{Prog(\mu, Z, Z')\} = \min\{\top\} = \top$$

Example

Consider parity game G:

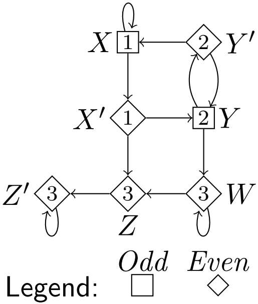


v	$\mu(v)$
X	$\top$
X'	$(0,1,0,0)$
Y	$(0,0,0,0)$
Y'	$(0,0,0,0)$
Z	$\top$
Z'	$\top$
W	$(0,0,0,0)$

$$Lift(\mu, W) = \min\{Prog(\mu, W, Z), Prog(\mu, W, W')\} = \min\{\top, (0,0,0,1)\} = (0,0,0,1)$$

Example

Consider parity game G:



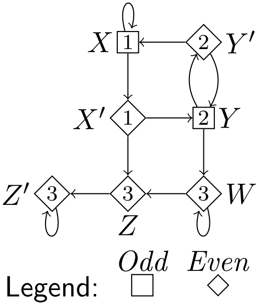
v	$\mu(v)$
X	$\top$
X'	$(0,1,0,0)$
Y	$(0,0,0,0)$
Y'	$(0,0,0,0)$
Z	$\top$
Z'	$\top$
W	$(0,0,0,1)$

Repeat lifting of W often

$$Lift(\mu, W) = \min\{Prog(\mu, W, Z), Prog(\mu, W, W')\} = \min\{\top, \top\} = \top$$

Example

Consider parity game G:

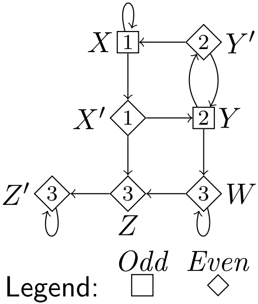


v	$\mu(v)$
X	$\top$
X'	$(0,1,0,0)$
Y	$(0,0,0,0)$
Y'	$(0,0,0,0)$
Z	$\top$
Z'	$\top$
W	$\top$

$$Lift(\mu, Y) = \max\{Prog(\mu, Y, W), Prog(\mu, Y, Y')\} = \max\{\top, (0,0,0,0)\} = \top$$

Example

Consider parity game G:

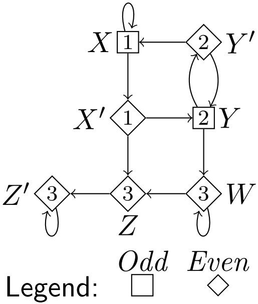


v	$\mu(v)$
X	$\top$
X'	$(0,1,0,0)$
Y	$\top$
Y'	$(0,0,0,0)$
Z	$\top$
Z'	$\top$
W	$\top$

$$Lift(\mu, X') = \min\{Prog(\mu, X', Z), Prog(\mu, X', Y)\} = \min\{\top, \top\} = \top$$

Example

Consider parity game G:

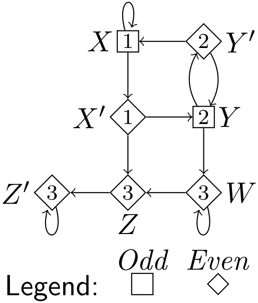


v	$\mu(v)$
X	$\top$
X'	$\top$
Y	$\top$
Y'	$(0,0,0,0)$
Z	$\top$
Z'	$\top$
W	$\top$

$$\mathit{Lift}(\mu, X') = \min\{\mathit{Prog}(\mu, Y', X), \mathit{Prog}(\mu, Y', Y)\} = \min\{\top, \top\} = \top$$

Example

Consider parity game G:



v	$\mu(v)$
X	$\top$
X'	$\top$
Y	$\top$
Y'	$\top$
Z	$\top$
Z'	$\top$
W	$\top$

μ is least game parity progress measure, and $\| \mu \| = \{v \mid v \in V \wedge \mu(v) \neq \top\} = \emptyset$ is winning set for player $Even$. Hence player Odd wins from all vertices

Complexity

Let $G = (V, E, p, (V_{Even}, V_{Odd})$ be a parity game;
$n = |V|, e = |E|, d = \max\{p(v) \mid v \in V\}$.

Worst-case running time complexity:

$$\mathcal{O}(de \cdot (\frac{n}{\lfloor d/2 \rfloor})^{\lfloor d/2 \rfloor})$$

Lowerbound on worst-case:

$$\Omega((\lceil n/d \rceil)^{\lceil d/2 \rceil})$$

Complexity

Let $G = (V, E, p, (V_{Even}, V_{Odd})$ be a parity game;
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- Algorithm with best known upper bound: Big step algorithm due to Schewe, with complexity

$$\mathcal{O}(d \cdot n^{d/3})$$

- Big step combines recursive algorithm with small progress measures;

Exercise

Consider the following parity game:

- Compute the winning sets W_{Even}, W_{Odd} for players $Even$ and Odd in this parity game using the small progress measures algorithm.
- Compare the solution with the solution obtained using the recursive algorithm and Gauß elimination.