Algorithms for Model Checking (2IW55)

Lecture 8

Boolean Equation Systems

Background material: Chapter 3 and 6 of
A. Mader, "Verification of Modal Properties using Boolean Equation Systems", Ph.D. thesis, 1997

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Outline

Boolean Equation Systems

- Boolean Equation Systems are a versatile formal framework for verification.
- Boolean Equation Systems are systems of fixed point equations.

Given a set Var of propositional variables. A Boolean Expression is defined by:

$$f ::= X \mid \text{true} \mid \text{false} \mid f \wedge f \mid f \vee f$$

A Boolean Equation is an equation of the form $\sigma X = f$, where $X \in Var$, $\sigma \in \{\mu, \nu\}$ and f is a Boolean Expression. A Boolean Equation System is a sequence of Boolean Equations:

$$\mathcal{E} ::= \varepsilon \mid (\sigma X = f)\ \mathcal{E}$$

Note:

- Negation is not allowed, in order to ensure monotonicity.
- The order of equations is important. Intuitively, the topmost sign has priority.

Boolean Equation Systems

- A variable W that occurs in a Boolean Expression of a BES $\mathcal{E}$ is called bound, if there is an equation for W in $\mathcal{E}$, otherwise W is called free.
- If propositional variables are bound uniquely (i.e., at most once), the BES is well-formed; we only consider well-formed BESs.
- If $\mathcal{E}$ contains no free variables, $\mathcal{E}$ is closed, otherwise it is open.

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Example

An example of a closed BES $\mathcal{E}$ with three propositional variables X, Y and Z:

$$(\mu X = (X \land Y) \lor Z)\ (\nu Y = X \land Y)\ (\mu Z = Z \land X)$$

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An example of a closed BES $\mathcal{E}$ with three propositional variables X, Y and Z:

$$(\mu X = (X \wedge Y) \vee Z)\ (\nu Y = X \wedge Y)\ (\mu Z = Z \wedge X)$$

An example of an open BES $\mathcal{F}$ with three propositional variables X, Y and Z:

$$(\mu X = Y \vee Z)\ (\nu Y = X \wedge Y)$$

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An example of a BES that is not well-formed:

$$(\mu X = X)\ (\nu X = X)$$

Boolean Equation Systems

Intuitive semantics:

- Let Val be the set of all functions $\eta : Var \to \{\mathsf{false}, \mathsf{true}\}$
- The solution of a BES is a valuation: $\eta : Val$
- Let $[f](\eta)$ denote the value of boolean expression f under valuation η.
- For the solution η of a BES $\mathcal{E}$, we wish $\eta(X) = [f](\eta)$ for all equations $\sigma X = f$ in $\mathcal{E}$.
- Also, we want the smallest (for μ) or greatest (for ν) solution, where topmost equation signs take priority over equation signs that follow.

Given a BES $\mathcal{E}$, we define $[\![\mathcal{E}]\!] : Val \to Val$ by recursion on $\mathcal{E}$.

$$\begin{cases} [\![\varepsilon]\!](\eta) & := \eta \\ [\![(\mu X = f)\ \mathcal{E}]\!](\eta) & := [\![\mathcal{E}]\!](\eta[X := [f](\eta_\mu)]) \text{ where } \eta_\mu := [\![\mathcal{E}]\!](\eta[X := \mathsf{false}]) \\ [\![(\nu X = f)\ \mathcal{E}]\!](\eta) & := [\![\mathcal{E}]\!](\eta[X := [f](\eta_\mu)]) \text{ where } \eta_\nu := [\![\mathcal{E}]\!](\eta[X := \mathsf{true}]) \end{cases}$$

Outline

Model Checking using BESs

Transformation of the μ-calculus model checking problem to BES

- Given is the following model checking problem: $M, s \models \sigma X.\ f$
  - a closed μ-calculus formula $\sigma X.\ f$ in Positive Normal Form and,
  - a Mixed Kripke Structure $M = \langle S, s_0, \mathit{Act}, R, L \rangle$.
  - $s \in S$ is a state
- We define a BES $\mathcal{E}$ with the following property:

$$([\![\mathcal{E}]\!](\eta))(X_s) = \text{true iff } M, s \models \sigma X.\ f$$

  i.e. formula $\sigma X.\ f$ holds in state s if and only if the solution for X_s yields true.

- This BES is defined as follows:
  - For each subformula $\sigma' Y.g$ and for each state $s \in S$, we add the following equation:

$$\sigma' Y_s \ = \ RHS(s, g)$$

  - Important: The order of the equations respects the subterm ordering in the original formula $\sigma X.\ f$.

Model Checking using BESs

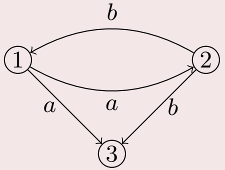
The Right-Hand Side of an equation is defined inductively on the structure of the μ-calculus formula:

$$
\begin{array}{lcl}
RHS(s,p) & = & p \in L(s) \\
RHS(s,X) & = & X_s \\
\\
RHS(s, f \wedge g) & = & RHS(s,f) \wedge RHS(s,g) \\
RHS(s, f \vee g) & = & RHS(s,f) \vee RHS(s,g) \\
\\
RHS(s,[a]f) & = & \bigwedge_{t \in S} \{RHS(t,f) \mid s \xrightarrow{a} t\} \\
RHS(s,\langle a\rangle f) & = & \bigvee_{t \in S} \{RHS(t,f) \mid s \xrightarrow{a} t\} \\
\\
RHS(s,\mu X.\ f) & = & X_s \\
RHS(s,\nu X.\ f) & = & X_s
\end{array}
$$

conventions: $\bigwedge_{t \in S} \emptyset =$ true and $\bigvee_{t \in S} \emptyset =$ false

Model Checking using BESs

Example

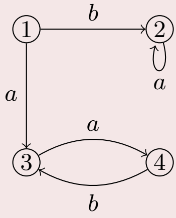


- $RHS(1,\ [a]X) = RHS(2,\ X) \wedge RHS(3,\ X) = X_2 \wedge X_3$.
- $RHS(2,\ \langle b\rangle Y) = RHS(1,\ Y) \vee RHS(3,\ Y) = Y_1 \vee Y_3$.
- $RHS(3, \langle b\rangle Y) =$ false (empty disjunction!)
- $$\begin{array}{rl} & RHS(1,\ [a]\langle b\rangle \mu Z.\ Z) \\ = & RHS(2,\ \langle b\rangle \mu Z.\ Z) \wedge RHS(3,\ \langle b\rangle \mu Z.\ Z)\wedge \\ = & (RHS(1,\ \mu Z.Z) \vee RHS(3,\ \mu Z.Z)) \wedge \text{false} \\ = & (Z_1 \vee Z_3) \wedge \text{false} \end{array}$$
- Translation of $\mu X.\langle b\rangle\text{true} \vee \langle a\rangle X$ to BES:

$$(\mu X_1 = X_3 \vee X_2)\ (\mu X_2 = \text{true})\ (\mu X_3 = \text{false})$$

Model Checking using BESs

Example



μ-calculus formula: $\nu X.\big([a]X \wedge \nu Y.\mu Z.(\langle b\rangle Y \vee \langle a\rangle Z)\big)$
Translates to the following BES:

$$
\begin{array}{lcl}
\nu X_1 & = & X_3 \wedge Y_1 \\
\nu X_2 & = & X_2 \wedge Y_2 \\
\nu X_3 & = & X_4 \wedge Y_3 \\
\nu X_4 & = & \mathsf{true} \wedge Y_4 \\
\nu Y_1 & = & Z_1 \\
\nu Y_2 & = & Z_2 \\
\nu Y_3 & = & Z_3 \\
\nu Y_4 & = & Z_4 \\
\mu Z_1 & = & Y_2 \vee Z_3 \\
\mu Z_2 & = & \mathsf{false} \vee Z_2 \\
\mu Z_3 & = & \mathsf{false} \vee Z_4 \\
\mu Z_4 & = & Y_3 \vee \mathsf{false}
\end{array}
$$

Outline

Solving BESs

- We reduced the model checking problem $M, s \models f$ to the solution of a BES with $\mathcal{O}(|M| \times |f|)$ equations.
- We now want a fast procedure to solve such BESs.
- An extremely tedious way to solve a BES is to unfold its semantics.
- A very appealing solution is to solve it by Gauß Elimination.

Solving BESs

Gauß Elimination uses the following 4 basic operations to solve a BES:

- local solution: eliminate X in its defining equation:

$$\mathcal{E}_0\ (\mu X = f)\ \mathcal{E}_1 \quad \text{becomes} \quad \mathcal{E}_0\ (\mu X = f[X := \mathsf{false}])\ \mathcal{E}_1$$
$$\mathcal{E}_0\ (\nu X = f)\ \mathcal{E}_1 \quad \text{becomes} \quad \mathcal{E}_0\ (\nu X = f[X := \mathsf{true}])\ \mathcal{E}_1$$

- Substitute definitions to the left:

$$\mathcal{E}_0\ (\sigma_1 X = X \vee Y)\ \mathcal{E}_1\ (\sigma_2 Y = Y \wedge X)\ \mathcal{E}_2$$
$$\text{becomes:} \quad \mathcal{E}_0\ (\sigma_1 X = X \vee (Y \wedge X))\ \mathcal{E}_1\ (\sigma_2 Y = Y \wedge X)\ \mathcal{E}_2$$

- Substitute closed equations to the right:

$$\mathcal{E}_0\ (\sigma_1 X = \mathsf{true})\ \mathcal{E}_1\ (\sigma_2 Y = Y \wedge X)\mathcal{E}_2$$
$$\text{becomes:} \quad \mathcal{E}_0\ (\sigma_1 X = \mathsf{true})\ \mathcal{E}_1\ (\sigma_2 Y = Y \wedge \mathsf{true})\ \mathcal{E}_2$$

- Boolean simplication: At least the following:

$$b \wedge \mathsf{true} \rightarrow b \qquad b \vee \mathsf{true} \rightarrow \mathsf{true} \qquad b \wedge \mathsf{false} \rightarrow \mathsf{false} \qquad b \vee \mathsf{false} \rightarrow b$$

Solving BESs

Example

$(\mu X = X \vee Y)\ (\nu Y = X \vee (Y \wedge Z))\ (\mu Z = Y \wedge Z)$

local $\rightarrow$

$(\mu X = \text{false} \vee Y)\ (\nu Y = X \vee (\text{true} \wedge Z))\ (\mu Z = Y \wedge \text{false})$

simplifications $\rightarrow$

$(\mu X = Y)\ (\nu Y = X \vee Z))\ (\mu Z = \text{false})$

substitution backwards $\rightarrow$

$(\mu X = Y)\ (\nu Y = X \vee \text{false})\ (\mu Z = \text{false})$

simplifications $\rightarrow$

$(\mu X = Y)\ (\nu Y = X)\ (\mu Z = \text{false})$

substitution backwards $\rightarrow$

$(\mu X = X)\ (\nu Y = X)\ (\mu Z = \text{false})$

local $\rightarrow$

$(\mu X = \text{false})\ (\nu Y = X)\ (\mu Z = \text{false})$

substitution to the right $\rightarrow$

$(\mu X = \text{false})\ (\nu Y = \text{false})\ (\mu Z = \text{false})$

Solving BESs

Gauß Elimination is a decision procedure for computing the solution to a BES.

Input: a BES $(\sigma_1 X_1 = f_1) \ \ldots \ (\sigma_n X_n = f_n)$. Returns: the solution for X_1.

for $i = n$ **downto** 1 **do**
  if $\sigma_i = \mu$ **then** $f_i := f_i[X_i := \text{false}]$
  else $f_i := f_i[X_i := \text{true}]$
  end if
  for $j = 1$ **to** $i - 1$ **do** $f_j := f_j[X_i := f_i]$
  end for
end for

Note:

- Invariants of the outer loop:
  - f_i contains only variables X_j with $j \leq i$.
  - for all $i < j \leq n$, X_j does not occur in f_j.
- Upon termination $(i = 0)$, $\sigma_1 X_1 = f_1$ is closed and evaluates to true or false.
- One could substitute the solution for X_1 to the right and repeat the procedure to solve X_2, *etcetera*.

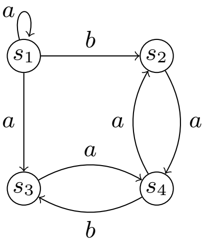
Solving BESs

Complexity of Gauß Elimination.

- Note that in $\mathcal{O}(n^2)$ substitutions, we obtain the final answer for X_1.
- However, f_1 can have $\mathcal{O}(2^n)$ different copies of e_n as subterms, so intermediate expressions could become exponentially big.
- Practical efficiency increases a lot if one keeps all intermediate terms simplified all the time.
- Gauß Elimination can be sped up if a forward dependency analysis is conducted (so-called local model checking).
- Precise efficiency depends heavily on the set of simplification rules.
- Precise complexity of Gauß Elimination is yet unknown.
- Complexity of Gauß Elimination is independent of the alternation depth (see Proposition 6.4 [Mader]).

Outline

Exercise



Consider the following μ-Calculus formula f:

$$\nu X.\big([a]X \wedge \nu Y.\mu Z.(\langle b\rangle Y \vee \langle a\rangle Z)\big)$$

- Use the Emerson-Lei algorithm for computing whether $M, s_1 \models f$.
- Translate the model checking question $M \models f$ to a BES; indicate how $M, s \models \phi$ corresponds to the variables in the BES.
- Solve the BES by Gauß Elimination.