2IV60 – Exercise 6: Illumination and shading

1. Given a triangle after transformation, with reflection properties and a light source. Give three ways to calculate colors for the interior of the triangle and discuss pro's and cons.
   In telegram style, for more detail see slides and book: Flat shading (constant color), Gouraud shading (calculate color per vertex and interpolate colors), Phong shading (calculate normal per vertex, interpolate normals, apply shading model per pixel). The generic trade-off is speed versus quality.

2. In an application for the visualization of hard disks (SequoiaView) files are shown as cushions. These cushions are modeled as quadratic surfaces:
   $z(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$,
   where x and y are given in pixels and where z represents the height of the cushion. Furthermore, a light source is given, which is located at infinity in the direction $L = (L_x, L_y, L_z)$ with intensity I. The surface reflects the light diffusely and has reflection coefficient k.

   a) Calculate the normal $N(x, y)$ on the cushion surface;
   This can be done in two ways. We can consider the surface as a parametric surface $P(u,v) = (u, v, z(u, v))$, calculate partial derivatives of P to u and v, and take the cross-product. That approach is somewhat involved here. A much faster route is to define the surface implicitly:
   $F(x, y, z) = z - (ax^2 + bxy + cy^2 + dx + ey + f) = 0$
   and calculate the gradient on the surface:
   $N(x, y) = \nabla F(x, y, z) = (-(2ax + by + d), -(bx + 2cy + e), 1)$.

   b) Geef een procedure om de helderheid $I_s = I_s(x, y)$ te berekenen;
   The standard formula for this is:
   $$\begin{aligned} I_s(x, y) &= kL \cdot N / |N| \\ &= kL \cdot (A, B, 1) / \sqrt{A^2 + B^2 + 1} \\ &= k(L_x A, L_y B, L_z) / \sqrt{A^2 + B^2 + 1} \end{aligned}$$
   with $A = -(2ax + by + d)$ and $B = -(bx + 2cy + e)$.
   This is the standard model for diffuse reflection, based on Lambert's law. The fact that the normal always points towards the viewer is used, i.e., the z component is always 1, so no test for negative dot products is needed here. Furthermore, the assumption is made that L is normalized already, which can easily be done initially. The normalization of N has to be done per pixel, however.

   c) A colleague claims that the use of $L = (0,0,1)$ reduces the number of calculations strongly. Is that correct?
   Well, the gain is not that large. It saves two multiplications and an addition, but the normalization of N is much more involved. On the other hand, it gives rise to a big loss: the image quality degrades. A light source shining from an oblique angle gives a

much more natural and attractive image, and boundaries between cushions are much better visible.

3. Suppose, we want to enable users of an interactive graphics application to specify the position of a light source easily. We solve this by showing a sphere with a radius of r pixels in a separate window, projected orthogonally, centered at a pixel (c, c). Users can indicate where they want a *highlight* (i.e., the specular reflection of the light source) to appear by clicking at a point of the sphere with the mouse.

   a) Give a procedure to determine the normal on the surface of the sphere, given a point (x, y) indicated by the user.

   The 2D indication of the user can be viewed as the specification of a line in 3D, perpendicular to the screen. This line (or ray) has starting point $(x,y,0)$ and direction $(0,0,1)$. Next we determine the intersection of this line with the sphere. The sphere is defined by $(x-c)^2+(y-c)^2+z^2-r^2=0$. The x- and y-coordinate are given, the z-coordinate can easily be found with: $z=\sqrt{r^2-(x-c)^2+(y-c)^2}$. If the term below the square root sign is negative, then the sphere is not hit and no action has to be taken. Given a point P on a sphere with center C, the normal at point P has the direction P-C. The normal vector N hence is given by:
   $((x-c),(y-c),\sqrt{r^2-(x-c)^2+(y-c)^2})$.

   b) Determine the corresponding position of the light source (at infinity).

   Pure specular reflection is driven by the law that the angle of incidence and exit are equal. Here it holds that $V+R=2(N\cdot V)N$, where V is a vector in the direction of the incoming ray (in the direction of the eye) and R is the reflected ray (in the direction of the light source). We find R by rewriting the equation and substituting V=(0,0,1):

   $$\begin{aligned} R &= 2(N\cdot V)N-V \\ &= 2z((x-c),(y-c),z)-(0,0,1). \end{aligned}$$

4. Given two polygons P and Q, with colors A and B, and with transparency α en β. The background color is C.

   a) Give the colors of the two polygons after display, assuming no overlap, using a simple model for transparency.

   Define K_P as the color of the interior of polygon P. A simple model for transparency is by blending foreground and background color, weighted by the transparency. This gives here:

   $K_P=(1-\alpha)A+\alpha C$, and
   $K_Q=(1-\beta)B+\beta C$.

   b) Give the color that results when polygon P is shown on top of polygon Q, and the color that we get when Q is displayed on top of P;

   Define K_{PQ} as the color we get if polygon P is shown first and polygon Q afterwards. We get:

$$
\begin{aligned}
K_{PQ} &= (1-\beta)B + \beta K_P \\
&= (1-\beta)B + \beta((1-\alpha)A + \alpha C) \\
&= (1-\alpha)\beta A + (1-\beta)B + \alpha\beta C, \\
K_{QP} &= (1-\alpha)A + \alpha K_Q \\
&= (1-\alpha)A + \alpha((1-\beta)B + \beta C) \\
&= (1-\alpha)A + \alpha(1-\beta)B + \alpha\beta C.
\end{aligned}
$$

c) For which value(s) of α en β does the order of display not matter for the final result?

The order does not matter if $K_{PQ} = K_{QP}$. After substitution and simplification we get $(1-\alpha)(1-\beta)(B-A) = 0$. This condition is satisfied only if $\alpha = 1$ or $\beta = 1$, (hence, if one of the polygons is fully transparent, also known as invisible), or if B=A (hence, both polygons have the same color). Hence, for instance when the polygons have different colors but are equally transparent, order does matter.

5. Given a cylinder: $x^2 + y^2 \leq 1;\ 0 \leq z \leq h.$ It is desired to make an image of the cylinder using ray-tracing. The ray is given by a point *P* and a vector *V*. Determine:

a) The intersections of the ray with the cylinder wall;

A point *Q*(*t*) on the ray is given by *Q*(*t*) = *P* + *Vt*. If we substitute this in the equation of the cylinder wall, we get

$(P_x + V_x t)^2 + (P_y + V_y t)^2 = 1$ or

$(V_x^2 + V_y^2)t^2 + 2(P_x V_x + P_y V_y)t + (P_x^2 + P_y^2 - 1) = 0$ or

$At^2 + 2Bt + C = 0$ met $A = (V_x^2 + V_y^2)$, $B = (P_x V_x + P_y V_y)$ en $C = (P_x^2 + P_y^2 - 1)$.

If the discriminant $D = B^2 - AC$ is smaller than 0, then there is no intersection with the wall, else we find the intersections with $t_{1,2} = (-B \pm D)/A$. Substitution in the definition of the ray gives $Q_{1,2} = P + Vt_{1,2}$. Next we test if the *z*-coordinates of these points satisfy $0 \leq Q_{1,2;z} \leq h$, if not, then they are outside the wall. A special case is *A*=0: here the ray is parallel to the wall.

b) The intersections of the ray with the upper and lower plane of the cylinder;

For the intersection of the ray with the plane *z*=0 it holds that $P_z + V_z t_o = 0$, hence $t_o = -P_z / V_z$. The intersection point *Q* is found by substituting this value for *t* in the equation of the ray. For the intersection point has to hold that $Q_x^2 + Q_y^2 \leq 1$, otherwise the intersection point is outside the circle of the lower plane. The intersection point with the plane *z*=*h* can be found in the same way, here it holds that $P_z + V_z t_b = h$, hence $t_b = (h - P_z)/V_z$.

A special case here is $V_z = 0$, where the ray is parallel to the planes. Either the ray is completely in a plane if $P_z = 0$ or $P_z = h$ and completely outside in the other cases.

c) The interval where the ray is inside the cylinder.

We have found all valid intersection points, we only have to sort these with increasing t to get the interval where the ray is inside the cylinder. An alternative approach is to skip the additional tests in a) and b) ($0 \le z \le h$ for the points on the cylinder wall, inside circle tests for the intersection points with the planes) and to work directly with the intersection points with the infinite cylinder and infinite planes. We consider first the cylinder wall with intersection points t_1 and t_2. The ray is then inside the wall for $t \in [\min(t_1, t_2), \max(t_1, t_2)]$. For the lower and upper plane it holds that the ray is between both planes for $t \in [\min(t_o, t_b), \max(t_o, t_b)]$. We are interested in the interval where the ray is inside both intervals simultaneously, hence the interval $t \in [\min(t_1, t_2), \max(t_1, t_2)] \cap [\min(t_o, t_b), \max(t_o, t_b)]$. This resulting interval (possibly empty) can be determined easily.