Vertex-coloring

Consider a connected graph \( G = (V, E) \). Let \( \Delta(G) := \max_v \deg(v) \) denote the maximum degree, and let \( \chi(G) \) denote the vertex coloring number of \( G \). We make the following observations:

1. \( \chi(G) \leq \Delta(G) + 1 \). We can color the vertices of the graph with ‘colors’ \( 1, \ldots, \Delta + 1 \) as follows: take any labeling \( v_1, \ldots, v_n \), for \( i = 1, \ldots, n \) assign \( v_i \) the lowest number not used for its neighbors among \( v_1, \ldots, v_{i-1} \). As there are at most \( \Delta \) such neighbors, a color from \( 1, \ldots, \Delta + 1 \) is always available.

2. If \( G \) contains a vertex \( v \) of degree \( \deg(v) < \Delta(G) \), then \( \chi(G) \leq \Delta(G) \). To find the coloring with \( \Delta \) colors, make a breadth first search of the graph, starting at node \( v \). Label the vertices \( v_1, \ldots, v_n \), where \( v_i \) is the \( i \)th vertex encountered. Hence \( v_1 = v \). Now color the nodes in backward order. For each vertex \( v_i \) at most \( \Delta - 1 \) neighbors are among \( v_{i+1}, \ldots, v_n \) that receive a color before \( v_i \), hence there is a color available for \( v_i \). Note that for \( v_i \) there are \( \Delta - \deg(v) \) colors available.

3. (Brooks’ theorem) If \( G \) is not \( K_m \) or \( C_{2m+1} \) for some \( m \), \( \chi(G) \leq \Delta(G) \). A constructive proof is given below.

Proof of Brooks’ theorem.

1. If \( \Delta = 0 \), \( G = K_1 \); if \( \Delta = 1 \), \( G = K_2 \); if \( \Delta = 2 \), then \( G \) is a path, or an even cycle, and \( \chi = 2 \), or \( G \) is an odd cycle with \( \chi = 3 \). We may assume for the remainder that \( \Delta \geq 3 \).

2. If \( G \) has a one-node cut-set \( \{v\} \), we consider the sub-graphs \( G_1 \) and \( G_2 \). Both can be colored with \( \Delta(G) \) colors, as \( v \) has degree at most \( \Delta(G) - 1 \) in both \( G_1 \) and \( G_2 \).

3. If \( G \) has a two-node cut-set \( \{u, v\} \), with \( \{u, v\} \not\in E \), then we can again color both sub-graphs \( G_1 \) and \( G_2 \) with \( \Delta(G) \) colors as both vertices have degree at most \( \Delta(G) - 1 \), both in \( G_1 \) and in \( G_2 \). If in both graphs \( G_1 \) and \( G_2 \), at last one vertex \( u \) or \( v \) has degree at most \( \Delta(G) - 2 \), then both graphs can be colored with \( u \) and \( v \) in different colors, and the colorings can be combined to a coloring of \( G \) with \( \Delta(G) \) colors. If \( \deg(u) = \deg(v) = \Delta(G) - 1 \), then in \( G_2 \) they have degree 1; say \( uu' \in E, vv' \in E \), with \( u' \neq v' \), for some \( u', v' \in V(G_2) \). In this case replace the vertex cut-set by \( \{u, v'\} \). That one does the job.

4. If the above does not apply, then take any node \( w \in V \) of maximum degree. It must have two neighbors \( u, v \) say, such that \( uv \not\in E \), otherwise \( G = K_n \). The graph \( G - u - v \) is connected, and so its vertices can be labeled \( v_1, \ldots, v_{n-2} \) by breadth first search starting from \( w \) setting \( w = v_1 \). Now set \( u = v_{n-1} \), and \( v = v_n \), and color the vertices in order \( v_n, v_{n-1}, \ldots, v_1 \). For each vertex a color from \( 1, \ldots, \Delta \) is available.