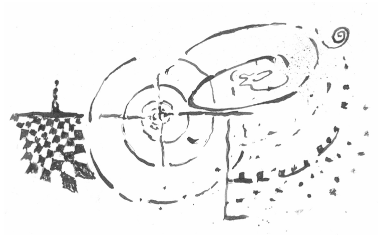
Algorithmic Adventures

From Knowledge to Magic



Book by Juraj Hromkovič, ETH Zurich
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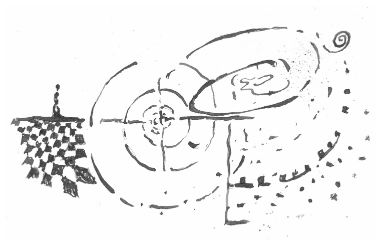
Quotation

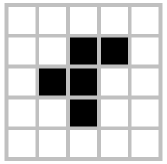


There is no greater loss than time which has been wasted

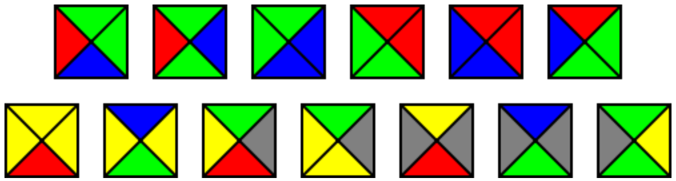
Michelangelo Buonarroti

Undecidability Is Not Rare

- Decide* whether a Game of Life configuration stabilizes



- Decide whether a set of Wang tiles can tile the plane



- Decide whether a Diophantine equation (multivariable polynomial equation, like $a^3 + b^3 = c^3$) has a solution in integers

- Decide whether a program has a specific non-trivial property, like whether it always halts, always outputs 0, ... [cf. Rice's Theorem]

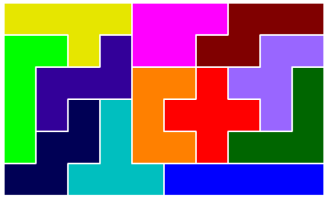
*In each case, the algorithm needs to work for *all* possible inputs (shown in yellow). All these decision problems turn out to involve a *universal* mechanism.

Some Algorithms Are Very Inefficient

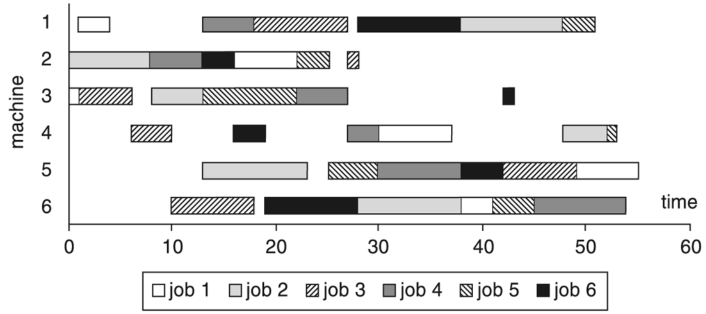
For some *algorithmically solvable* problems, our algorithmic solutions turn out to be very slow

Slow algorithms are *practically* unusable:

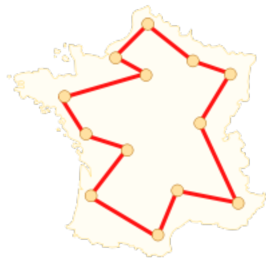
- Packing puzzles



- Scheduling jobs on machines



- Traveling Salesman Problem (**TSP**): find shortest tour visiting each town in a given set, given their distances



How can we investigate this phenomenon?

How can we overcome this limitation?

Algorithmic Complexity

The **time complexity** of algorithm A on input I:

number of instructions performed in computation of A on I

The **space complexity** of algorithm A on input I:

amount of memory used in computation of A on I

Complexity varies with **size of the input** (amount of input data)

The **time complexity** of algorithm A **as function of input size**:

$Time_A(n)$ = *worst-case* number of instructions performed in computation of A on any input of size n

Asymptotic Algorithmic Time Complexity

The function $\mathit{Time}_A(n)$ also depends on details of the programming language and implementation of the algorithm as program

Definition Function $f(n) \geq 0$ is $\mathcal{O}(g(n))$ ('f is big oh of g') when

$$f(n) \leq C \cdot g(n) \text{ for some constant } C \text{ and all sufficiently large } n$$

Example: $10n^2 + 7n + 20$ is $\mathcal{O}\left(n^2\right)$, but not $\mathcal{O}(n)$ and not $\mathcal{O}(\log n)$

The **asymptotic time complexity** of algorithm A is $f(n)$:

$$\mathit{Time}_A(n) \text{ is } \mathcal{O}(f(n)) \quad \text{and} \quad f(n) \text{ is } \mathcal{O}(\mathit{Time}_A(n))$$

The asymptotic complexity is *robust*, independent of implementation

Complexity classes: Constant, Logarithmic, Linear, Linearithmic $\mathcal{O}(n \cdot \log n)$, Quadratic, Cubic, ..., Polynomial, Exponential, ...

Asymptotic Time Complexity Examples

Complexity	Name	Example*
$\mathcal{O}(1)$	Constant	Determine whether n-bit number is even
$\mathcal{O}(\log n)$	Logarithmic	Find item in sorted list by *Binary Search*
$\mathcal{O}(n)$	Linear	Find item in list by *Linear Search*
$\mathcal{O}(n \log n)$	Linearithmic	Sort list by *Merge Sort*
$\mathcal{O}\left(n^2\right)$	Quadratic	Sort list by *Bubble Sort*
$\mathcal{O}\left(n^k\right)$	Polynomial	Determine whether n-bit number is *prime*
$\mathcal{O}(2^n)$	Exponential	Solve TSP by *Dynamic Programming*
$\mathcal{O}(n!)$	Factorial	Solve TSP by *Brute Force Search*

*The input is a list of n elements (possibly bits)

What Is the Limit of Practical Solvability?

n / $f(n)$	10	50	100	300
$10n$	100	500	1000	3000
$2n^2$	200	5 000	20 000	180 000
n^3	1000	125 000	1 000 000	27 000 000
2^n	1024	16 digits	31 digits	91 digits
$n!$	$\approx 3.6 \cdot 10^6$	65 digits	158 digits	615 digits

A problem is called **tractable** when it can be solved by a **polynomial** algorithm (asymptotic time complexity is $\mathcal{O}\left(n^k\right)$ for some constant k)

P denotes the class of all **polynomial decisions problems**

How Much More Can You Do on a 2× Faster Machine?

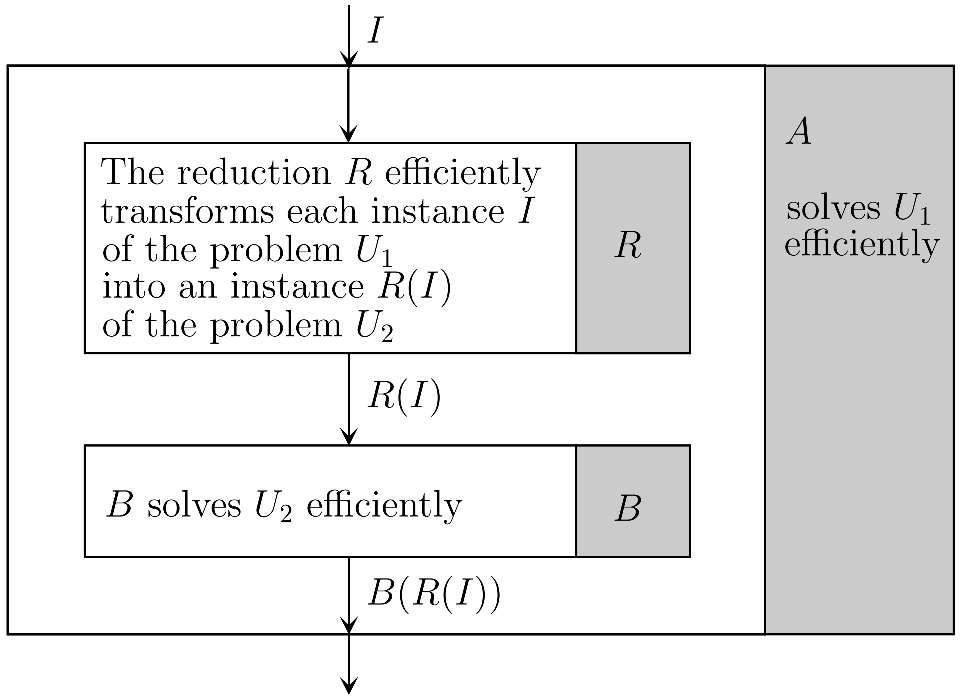
Assume $n = 100$ takes 1 hour on machine A.
How much further do you get on a 2× faster machine B in 1 hour?

	Time	n **on** A	n **on** B	**More on** B	**Factor**
Logaritmic	$C_1 \log_2 n$	100	10000	9900	100
Linear	$C_2\, n$	100	200	100	2
Linearitmic	$C_3\, n \log_2 n$	100	178	78	1.78
Quadratic	$C_4\, n^2$	100	141	41	1.41
Cubic	$C_5\, n^3$	100	126	26	1.26
Exponential	$C_6\, 2^n$	100	101	1	1.01

Polynomial-time Reduction

Algorithm R is a **polynomial reduction** from problem U_1 to U_2 when

- R is a *polynomial* algorithm, and
- the solution for instance I of problem U_1 equals the solution for instance $R(I)$ of problem U_2, for all instances I of U_1



Polynomial-time Reduction

By definition, the following statements are equivalent:

- Problem U_1 is **polynomial-time reducible** to problem U_2
- There exists a polynomial reduction R from U_1 to U_2
- $U_1 \leq_{\text{pol}} U_2$
- Problem U_1 is *polynomially no harder than* problem U_2

An example follows

Knapsack Problem

Subset Sum Problem, or (simplified) **Knapsack Problem**:

> For a given positive integer K and set S of items x with positive integer size $s(x)$, does there exists a subset T of S whose total size $\sum_{x \in T} s(x)$ equals K?

K is the size of the knapsack, S contains the items to pack, and s gives their sizes.

The question is whether the knapsack can be filled exactly with a suitable selection T of the items.

Example: item sizes 110, 90, 70, 50, 30, 30, 20, and $K = 150$

Settling Debts Problems

A group of friends lend each other money throughout the year. They carefully record each transaction. When Alice lends 10 euro to Bob, this is recorded as Alice $\xrightarrow{10}$ Bob.

At the end of the year they wish to settle all their debts. Money can be transferred between any *pair* of persons.

Problem variants:

- minimize the number of transfers
- minimize the total amount transferred
- minimize both

Reduce Knapsack to Settling Debts

Given an instance I for Knapsack, construct an instance $R(I)$ for Settling Debts: $|S|$ positive balances $s(x)$ for $x \in S$, and two negative balances $-K$ and $K - \sum_{x \in S} s(x)$. N.B. The total balance = 0.

+110	+90	+70	+50	+30	+30	+20
	−150			−250		

The instance $R(I)$ requires at least $|S|$ transfers to settle, since each positive balance needs an outgoing transfer. A settling of all debts for $R(I)$ with $|S|$ transfers exists if and only if there exists a subset T of S whose total size equals K, that is, when it solves I.

Thus: Knapsack $\leq_{\text{pol}}$ Settling Debts *in minimum number of transfers*

Using Polynomial-time Reducibility $U_1 \leq_{\text{pol}} U_2$

(Compare to *algorithmic* reducibility and its uses, in Ch. 4)

If we know $U_1 \leq_{\text{pol}} U_2$, then this can be used in two ways:

1. Polynomial solvability of U_2 implies polynomial solvability of U_1 (Note the order of U_2 and U_1 here)
2. If U_1 can*not* be solved by a polynomial algorithm, then U_2 can*not* be solved by a polynomial algorithm

Many problems for which we have not found polynomial algorithms are polynomially equally hard: $U_1 \leq_{\text{pol}} U_2$ and $U_2 \leq_{\text{pol}} U_1$

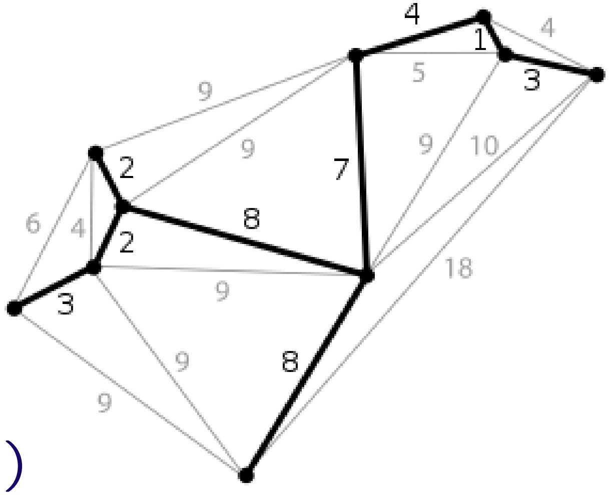
These problems are called **NP-hard**

Knapsack (Subset Sum) is known to be NP-hard
Hence, Settling Debts *in minimum number of transfers* is NP-hard

Easy/Hard Pairs

- *Hard*: Determine whether a graph has a **Hamiltonian circuit** that visits each **vertex** exactly once

  Easy: Determine whether a graph has an **Euler circuit** that visits each **edge** exactly once



- *Hard*: Determine a settling of all debts, that minimizes the number of transfers

  Easy: Determine a settling of all debts, that minimizes the total amount transferred

- *Hard*: **Traveling Salesman Problem** (TSP)

  Easy: Determine a **Minimum Spanning Tree** (MST) of a connected, edge-weighted graph: a set of edges of minimum total weight that connects all vertices (this is a *tree*; see figure)

Settling Debts, Minimizing Total Amount Transferred, Is Easy

Here is a greedy* algorithm:

1. Determine the balance b_i for each person

2. While there is still someone with a nonzero balance, do:

   (a) Select any person i with $b_i < 0$, and any person j with $b_j > 0$

   (b) Let m be the minimum of $-b_i$ and b_j; hence, $m > 0$

   (c) Include transfer $i \xrightarrow{m} j$ in the settlement

   (d) Increase b_i by m and decrease b_j by m

3. All $b_k = 0$, hence the included transfers settle all debts

*Step 2a makes it greedy: settle maximally among the first candidate pair found

Settling Debts, Minimizing Total Amount Transferred: Proof

$\sum_k b_k = 0$ holds initially and after every iteration of Step 2. (Invariant)

Step 2a is always possible, because $\sum_k b_k = 0$ and not all $b_k = 0$.

The repetition of Step 2 terminates, because in each iteration at least one nonzero b_k is reduced to zero by Step 2d.

Therefore, the number of transfers is at most N (number of persons). In fact, it is at most $N - 1$, because the final *two* nonzero balances cancel each other in a *single* transfer.

Let P be the total amount of the positive balances, and N the total amount of the negative balances. Hence, $P = -N$. The *minimum total amount to be transferred* equals P.

The total amount transferred equals P, and hence is minimal.

Summary

- Algorithmically solvable does not mean practically solvable
- **Time complexity** of an algorithm: how many steps it takes to compute an answer, in relation to input *size* (worst-case)
- **Complexity classes** defined in terms of *asymptotic* complexity: Polynomial time (**P**), Exponential time (**EXP**), ...
- **NP decision problem** ≈ YES answer *verifiable* in polynomial time
- **NP-hard**: class of *hardest* NP problems (**polynomial reduction**)
- **P $\stackrel{?}{=}$ NP**: Can all NP problems be solved in polynomial time?
- Today we know only *exponential* algorithms for NP-hard problems: **intractable**, practically unsolvable for larger inputs; not hopeless

Not Covered in the Slides

- It is hard to prove lower bounds on the (time) efficiency of algorithms that solve a specific problem
- For some problems, every algorithm solving it can be made more efficient; i.e., there is no lower bound on efficiency (**Blum's Speed-up Theorem**)
- The street supervision problem (VC = Vertex Cover)
- Approximation algorithm for VC